Math 113: Introduction to Abstract Algebra

Midterm March 15th, 2002

Weingart

Name: ______________________________

Signature: ___________________________

There are 9 problems on this midterm worth 100 points of 400 for the class in total. The first 5 problems are each worth 8 points for the correct answer, whereas the last 4 problems are more difficult and worth 15 points each. You must show your work to get any credit for the last 4 problems. Successful midterm!

1	2	3	4	5	6	7	8	9	Total

Problem 1: (8 points)
How many elements of the cyclic group $\mathbb{Z}_{15}$ have order 15?

- ☐ $\mathbb{Z}_{15}$ has 8 elements of order 15.
- ☐ $\mathbb{Z}_{15}$ has 10 elements of order 15.
- ☐ $\mathbb{Z}_{15}$ has 14 elements of order 15.

Problem 2: (8 points)
How many different homomorphisms $\phi: \mathbb{Z}_6 \longrightarrow \mathbb{Z}_4$ are there?

- ☐ There is only the trivial homomorphism from $\mathbb{Z}_6$ to $\mathbb{Z}_4$.
- ☐ There are exactly two different homomorphisms from $\mathbb{Z}_6$ to $\mathbb{Z}_4$.
- ☐ There are exactly four different homomorphisms from $\mathbb{Z}_6$ to $\mathbb{Z}_4$.

Problem 3: (8 points)
The symmetric group S_n, $n \geq 2$, acts on the cartesian product $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ by

$$\sigma \bullet (k, l) := (\sigma(k), \sigma(l))$$

for all $k, l \in \{1, 2, \ldots, n\}$. Is this action transitive?

- ☐ Yes, because S_n acts transitively on $\{1, 2, \ldots, n\}$.
- ☐ No, because this is no well–defined group action.
- ☐ No, because this action has exactly two orbits.

Problem 4: (8 points)
Cayley's Theorem says that every group G is isomorphic to a subgroup of a symmetric group S_n. But what precisely is this n for the alternating group A_4?

- ☐ Cayley's Theorem identifies A_4 with a subgroup of S_4.
- ☐ Cayley's Theorem identifies A_4 with a subgroup of S_8.
- ☐ Cayley's Theorem identifies A_4 with a subgroup of S_{12}.

Problem 5: (8 points)
Which of the following natural numbers is NOT the order of any element of S_7?

- ☐ 9 is not the order of any element of S_7.
- ☐ 10 is not the order of any element of S_7.
- ☐ 12 is not the order of any element of S_7.

Problem 6: (15 points)
Let H be a subset of a group G. In order to check whether H is a subgroup of G or not you have to verify three properties of H namely:

Problem 7: (15 points)
Consider the permutation $\sigma := (1,3,2) \circ (1,8,6) \circ (4,6,7,8)$ in S_8. What is its signature $\operatorname{sgn}\sigma$? Find the orbits of σ and calculate its order.

Problem 8: (15 points)
Consider a group G and the map $\phi : \; G \longrightarrow G, \; g \longmapsto g * g$. Prove that ϕ is a group homomorphism if and only if G is a commutative group.

Problem 9: (15 points)
One of the homework problems asked you to show that every $g \in G$ defines an automorphism $\mathrm{Ad}_g(x) := g * x * g^{-1}$ of G (which is trivial for a commutative group G of course). Find the kernel of the group homomorphism

$$\mathrm{Ad}: \; G \longrightarrow \mathrm{Aut}\, G, \quad g \longmapsto \mathrm{Ad}_g$$

mapping $g \in G$ to Ad_g and give this kernel an appropriate name.