Math 113, Introduction to Abstract Algebra (Kedlaya, fall 2002)
Second midterm exam, Wednesday, November 6, 2002

This is a closed-book exam. No notes, calculator, or other assistance are permitted.

There are five problems, each on a separate page, plus an extra page if you need room for scratch work. However, please show all your work on the problem pages; you may continue on the back if you need more space. Work on the scratch page will not be graded.

Problem 1. Give a one-sentence answer to each of the following questions. (5 points each)

(a) Define a normal subgroup of a group G.

(b) Describe the effect of conjugation by $(1,3)$ on a permutation of S_n, in terms of its cycle description.

(c) Define a zero divisor of a commutative ring R.

(d) Provide an example of a ring R with unity in which the set of units is not a subring of R.

Problem 2. Let the group S_4 act on the set X of ordered pairs of elements of $\{1,2,3,4\}$ (not necessarily distinct).

(a) Complete the following table by determining, for each permutation shown, the number of fixed points of that permutation acting on X. (10 points)

Permutation	Fixed points in X
(1)	
$(1,2)$	
$(1,2,3)$	
$(1,2)(3,4)$	
$(1,2,3,4)$	

(b) Use Burnside's formula to compute the number of orbits of the action of S_4 on X. (Hint: the sizes of the conjugacy classes of S_4, in the order listed in the table, are $1,6,8,3,6$.) (5 points)

Problem 3.

(a) For each prime p dividing the order of the alternating group A_5, list one Sylow p-subgroup of A_5. (10 points)

(b) For each prime p in (a), determine which possibilities for the number of Sylow p-subgroups contained in A_5 are consistent with the third Sylow theorem. (10 points)

Problem 4. Let G be the group $\mathbb{Z}_{10} \times \mathbb{Z}_{12}$.

(a) Find a subgroup H of G such that G/H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. (5 points)

(b) Find the order of the subgroup of G generated by $(5, 5)$. (5 points)

(c) Determine the number of homomorphisms ϕ from G to $\mathbb{Z}_2$. (Hint: any homomorphism ϕ is determined by its values on a set of generators of G.) (5 points)

Problem 5. Give careful proofs of the following statements. (10 points each)

(a) If ϕ is a homomorphism from G to G' and H is a subgroup of G, then the image $\phi[H]$ of H is a subgroup of G'.

(b) If a group G of order 27 acts on a set X of 32 elements, there must be at least one element of X fixed by all of G.

(c) If p is an odd prime number, then either $2^{(p-1)/2}+1$ or $2^{(p-1)/2}-1$ is a multiple of p.