MATH 113 – EXAM 1

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Time limit: 80 minutes

Name:

SID:

You may not consult any books or papers. You may not use a calculator or any other computing or graphing device other than your own head!

Unless instructed otherwise, you are required to justify all of your answers. An answer with no justification will receive little credit. Please write all answers in complete English sentences.

There are two extra blank pages at the end of the exam. You may use these for computations, but **I will not read them.** Please transfer all final answers to the page on which the question is posed.

GOOD LUCK!

Problem:	Your score:	Total points
1		25 points
2		15 points
3		12 points
4		15 points
5		18 points
6		15 points
Total:		100 points

1. **(25 points) True/False.** Please determine whether the following statements are true or false. Please justify your answer: give a proof or counterexample.

   (a) **(5 points)** Let m and n be positive integers with $gcd(m, n) = d$. Then $f : \mathbb{Z}_n \to \mathbb{Z}_m$ given by $f([x]_n) = [d \cdot x]_m$ is a well-defined function.

   (b) **(5 points)** $x \equiv 3 \pmod 5$ if and only if $2x \equiv 1 \pmod 5$.

(c) **(5 points)** In S_n, the composition of two cycles is again a cycle.

(d) **(5 points)** The map $f : \mathbb{Q}^\times \to \mathbb{Q}^\times$ given by $f(\frac{m}{n}) = \frac{n}{m}$ is a well-defined function.

(e) **(5 points)** Let A^{T} denote the transpose of the matrix A. The set

$$\left\{ A \in GL_n(\mathbb{R}) \,\middle|\, A^{-1} = A^{\mathrm{T}} \right\},$$

with the binary operation matrix multiplication, is a group.

You may use the fact that $(A \cdot B)^{\mathrm{T}} = B^{\mathrm{T}} \cdot A^{\mathrm{T}}$.

2. **(15 points)** Suppose that $\sigma \in S_n$ is a permutation, and write $\sigma = \tau_1 \cdots \tau_k$, where the τ_j's are disjoint cycles.

   Please prove that σ is an even permutation if and only if there is an even number of even-length cycles among the τ_j's.

3. **(12 points)** Let $(G, \cdot)$ be a group. Define a new binary operation $*$ on G by letting $g * h = h \cdot g$.

   Prove that $(G, *)$ is a group.

4. **(15 points)** For some $a \in \mathbb{R}$, define

$$G_a = \left\{ A \in GL_n(\mathbb{R}) \,\middle|\, det(A) = a \right\} \subseteq GL_n(\mathbb{R}).$$

Prove that G_a, with the operation matrix multiplication, is a group if and only if $a = 1$.

5. **(18 points)** Decide whether or not the following relations are equivalence relations. Justify your answer: please provide a proof or counter-example.
   (a) **(9 points)** $S = S_n$ and $\sigma \sim \tau$ if $\sigma(\tau^{-1})$ is an even permutation.

   (b) **(9 points)** $S = \mathbb{R} \times \mathbb{R}$ and $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 y_2 = x_2 y_1$.

6. **(15 points)** Suppose that m and n are positive integers, $m|n$ and a is any integer. Prove that

$$[a]_m = [a]_n \cup [a+m]_n \cup [a+2m]_n \cup \cdots \cup [a+n-m]_n.$$