University of California, Berkeley **Department of Mechanical Engineering**
ME 106 – Fluid Mechanics **Spring 2009**

Midterm 1

Answer all 3 questions. Hand in the answer sheets only.

QUESTION 1. Suppose that a scalar quantity ϕ, which describes some physical property of a fluid, is specified by

$$\phi = \phi_E(x, y, z, t) = \phi_L(x_0, y_0, z_0, t),$$

where the function ϕ_E is the Eulerian represention and the function ϕ_L is the Lagrangian representation.

(a) Define the material time derivative of ϕ and derive the formula

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \nabla\phi \cdot \boldsymbol{V},$$

where $\boldsymbol{V} = \boldsymbol{V}(x, y, z, t)$ is the velocity field of the fluid.

(b) Suppose that the density and velocity fields in a fluid are given by

$$\rho = \rho_0 + txy^2, \;\; \boldsymbol{V} = y\sin(2t)\boldsymbol{i} + z^2(\boldsymbol{j} + \boldsymbol{k}),$$

where $\rho_0(x_0, y_0, z_0)$ is the density of the fluid at time $t = 0$. Calculate $D\rho/Dt$.

QUESTION 2. For a fluid in motion, let

$$\boldsymbol{n}(x, y, z, t) = n_x\boldsymbol{i} + n_y\boldsymbol{j} = \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}$$

be the outward unit normal vector to a surface passing through the point (x, y, z) at time t. Recall the stress vector formula

$$\boldsymbol{t}(x, y, z, t, \boldsymbol{n}) = \boldsymbol{t}(x, y, z, t, \boldsymbol{i})\,n_x + \boldsymbol{t}(x, y, z, t, \boldsymbol{j})\,n_y.$$

(a) Describe in words and with a diagram what $\boldsymbol{t}(x, y, z, t, \boldsymbol{i})$ represents physically.

(b) If at some spatial point (x, y, z) and time t, the stress vectors $\boldsymbol{t}(x, y, z, t, \boldsymbol{i})$ and $\boldsymbol{t}(x, y, z, t, \boldsymbol{j})$ have the values

$$\boldsymbol{t}(x, y, z, t, \boldsymbol{i}) = [-200\,\boldsymbol{i} + 150\,\boldsymbol{j} + 80\,\boldsymbol{k}] \quad \text{KPa}\,, \quad \boldsymbol{t}(x, y, z, t, \boldsymbol{j}) = [150\,\boldsymbol{i} - 200\,\boldsymbol{j}] \quad \text{KPa}\,,$$

(i) calculate the three components of the stress vector acting on a surface that passes through (x, y, z) at time t and has an outward unit normal vector $\boldsymbol{n}$ that lies in the xy-plane and makes an angle of $\pi/6$ radians with the positive x-axis; ***(ii)*** calculate the normal stress N and the shearing stress τ for the surface in ***(i)***.

QUESTION 3. Consider an inviscid fluid with constant density, and suppose that it is free from body forces. Recall Euler's hydrodynamical equation

$$-\boldsymbol{\nabla} p = \rho \boldsymbol{a} \qquad \text{N/m}^3$$

(a) If the fluid is spinning with constant angular about the z-axis in a closed container (in a gravity-free environment), it is known that the acceleration field may be expressed as

$$\boldsymbol{a} = -\frac{c}{\rho}(x\boldsymbol{i} + y\boldsymbol{j}) \qquad \text{m/s}^2\,.$$

where c is a positive constant. Solve for the pressure field.

(b) If the pressure has a constant value p_0 along the z-axis, calculate its value on the cylindrical surface

$$x^2 + y^2 = 1 \qquad \text{m}^2\,.$$

1/a)

Definition: The time rate of change of a field ϕ as experienced by a moving particle of fluid

$$\frac{D\phi}{Dt} = \frac{\partial \phi_L}{\partial t} \text{ in Langrangian form}$$

$$\phi_E = \phi_E(x, y, z, t)$$
$$\phi_L = \phi_L(x_0, y_0, z_0, t)$$

$$\frac{D\phi(x,y,z,t)}{Dt} = \frac{\partial\phi(x,y,z,t)}{\partial t} + \frac{\partial\phi(x,y,z,t)}{\partial x}\frac{\partial x(x_0,y_0,z_0,t)}{\partial t} \ldots$$
$$\ldots + \frac{\partial\phi(x,y,z,t)}{\partial y}\frac{\partial y(x_0,y_0,z_0,t)}{\partial t} + \ldots$$
$$\ldots + \frac{\partial\phi(x,y,z,t)}{\partial z}\frac{\partial z(x_0,y_0,z_0,t)}{\partial t}$$

Noting $\frac{\partial x(x_0,y_0,z_0,t)}{\partial t} = \frac{Dx}{Dt}$ since $\frac{\partial x_0}{\partial t} = \frac{\partial y_0}{\partial t} = \frac{\partial z_0}{\partial t} = 0$

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\frac{Dx}{Dt} + \frac{\partial\phi}{\partial y}\frac{Dy}{Dt} + \frac{\partial\phi}{\partial z}\frac{Dz}{Dt}$$

Noting $u = \frac{Dx}{Dt}$ $v = \frac{Dy}{Dt}$ $w = \frac{Dz}{Dt}$

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}u + \frac{\partial\phi}{\partial y}v + \frac{\partial\phi}{\partial z}w$$

$$\underline{V} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \underline{\nabla}\phi = \begin{bmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial z \end{bmatrix}$$

So,

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \underline{\nabla}\phi \cdot \underline{V}$$

b)

$$\rho = \rho_0 + txy^2 \qquad \underline{V} = y\sin(2t)\,\underline{i} + z^2(\underline{j} + \underline{k})$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \underline{\nabla}\rho \cdot \underline{V}$$

$$= \frac{\partial \rho}{\partial t} + \begin{bmatrix} \partial\rho/\partial x \\ \partial\rho/\partial y \\ \partial\rho/\partial z \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$= xy^2 + \begin{bmatrix} ty^2 \\ 2txy \\ 0 \end{bmatrix} \cdot \begin{bmatrix} y\sin(2t) \\ z^2 \\ z^2 \end{bmatrix}$$

$$= xy^2 + ty^3\sin(2t) + 2txyz^2$$

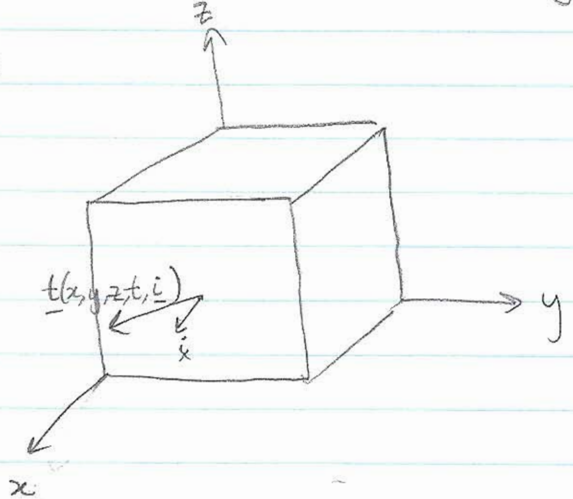
$$\boxed{\frac{D\rho}{Dt} = xy^2 + ty^3\sin(2t) + 2txyz^2}$$

2/ a) $\underline{t}(x,y,z;t;\underset{\sim}{i}) = \underline{\underline{T}}\,\underline{i}$, $\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\underline{t}(x,y,z,t;\underset{\sim}{i})$ represents the traction at position (x,y,z) and time t, on a surface with outward unit normal $\underset{\sim}{i}$.

Traction represents all surface forces acting on that surface.

On a cube:



b) $\underline{t}(x,y,z,t,\underset{\sim}{i}) = -200\underset{\sim}{i} + 150\underline{j} + 80\underline{k}$ [kPa]

$\underline{t}(x,y,z,t,\underline{j}) = 150\,\underline{i} - 200\underline{j}$ [kPa]

i) $\underline{t}(\underline{n}) = \underline{t}(\underline{i})n_x + \underline{t}(\underline{j})n_y$

$\underline{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$, $\theta = \frac{\pi}{6}$ $\Rightarrow$ $\underline{n} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \\ 0 \end{pmatrix}$

$\rightarrow t(n) = \begin{pmatrix} -200 \\ 150 \\ 80 \end{pmatrix}\sqrt{3}/2 + \begin{pmatrix} 150 \\ -200 \\ 0 \end{pmatrix}\frac{1}{2} \simeq \begin{pmatrix} -98.2 \\ 29.9 \\ 69.2 \end{pmatrix}$ [kPa]

2/b) ii) $N = \underline{t} \cdot \underline{n} = \begin{pmatrix} -98.2 \\ 29.9 \\ 69.2 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \\ 0 \end{pmatrix} = \underline{\underline{-70.1 \text{ kPa}}}$

$\|t\| = \left(98.2^2 + 29.9^2 + 69.2^2\right)^{\frac{1}{2}} = 123.8 \text{ kPa}$

$t^2 = N^2 + S^2$

$\Rightarrow \quad S = \left(t^2 - N^2\right)^{\frac{1}{2}} = \underline{\underline{102.1 \text{ kPa}}}$

3/ $-\nabla P = \rho \underline{a}$

a) $\underline{a} = -\frac{c}{\rho}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \Rightarrow -\nabla P = -c\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = -\begin{pmatrix} \partial P/\partial x \\ \partial P/\partial y \\ \partial P/\partial z \end{pmatrix}$

$$\Rightarrow \begin{cases} \frac{\partial P}{\partial x} = cx \\ \frac{\partial P}{\partial y} = cy \\ \frac{\partial P}{\partial z} = 0 \end{cases} \Rightarrow \begin{cases} P = \frac{c}{2}x^2 + f_1(y,z,t) \\ P = \frac{c}{2}y^2 + f_2(x,z,t) \end{cases}$$

Combining equations & collecting unknown consts of int as "k"

$\rightarrow P = \frac{c}{2}(x^2 + y^2) + k(t)$

b) $P(0,0,z) = \cancel{\frac{c}{2}(0+0)}^{\to 0} + k(t) = P_0 = \text{const}$

$\Rightarrow k(t) = k = P_0$

$\Rightarrow P(x,y,z) = \frac{c}{2}(x^2+y^2) + P_0$

On the cylindrical surface $x^2 + y^2 = 1$

$\rightarrow \underline{\underline{P = \frac{c}{2} + P_0}}$ Pa