Math 113 FINAL EXAM

May 13, 1996 Prof. Wu

1. (5%) Prove that for an integer n, $3 \mid n \Longleftrightarrow 3 \mid$ (sum of digits of n).

2. (5%) Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ be a polynomial with integer coefficients, and let r be a rational number such that $f(r) = 0$. Show that r has to be an integer and $r \mid a_0$.

3. (5%) Find a minimal polynomial of $\sqrt[3]{1+\sqrt{3}}$ over $\mathbb{Q}$. (Be sure to prove that it is minimal.)

4. (5%) Let n be a positve integer ≥ 2 such that $n \mid (b^{n-1} - 1)$ for all integers b which are not a multiple of n. What can you say about n?

5. (5%) Do the nonzero elements of $\mathbb{Z}_{13}$ form a cyclic group under multiplication? Give reasons.

6. (10%) Let p be a prime.

(a) Prove: $p \left| \binom{p}{k} \right.$ for $k = 1, \cdots p-1$, where $\binom{p}{k} \equiv \dfrac{p!}{k!(p-k)!}$.

(b) Prove: the mapping $f : \mathbb{Z}_p \to \mathbb{Z}_p$ defined by $f(k) = k^p$ for all $k \in \mathbb{Z}_p$ is a field isomorphism.

7. (10%) Is $x^4 + 2x + 3$ irreducible over $\mathbb{R}$? Is it irreducible over $\mathbb{Q}$? Give reasons.

8. (10%) Let $F \equiv \{a + ib : a,\ b \in \mathbb{Q}\}$ and let $K \equiv \mathbb{Q}[x]/(x^2+1)\mathbb{Q}[x]$. Show that F is isomorphic to K as fields by defining a map $\varphi : F \to K$ and show that φ has all the requisite properties.

9. (10%) If β is a root of $x^3 - x + 1$, find some $p(x) \in \mathbb{Q}[x]$ so that $(\beta^2 - 2)\,p(\beta) = 1$.

10. (10%) Let $\zeta = e^{i2\pi/3}$. Compute $(\mathbb{Q}(\zeta, \sqrt[3]{5}) : \mathbb{Q}(\zeta))$.

11. (25%) (In (a)–(d) below, each part could be done independently.)

(a) Assume that if p is a prime, then $x^{p-1} + x^{p-2} + \cdots + 1$ is irreducible over $\mathbb{Q}$. Compute $(\mathbb{Q}(\cos(2\pi/7) + i\sin(2\pi/7)) : \mathbb{Q})$. (Each step should be clearly explained.)

(b) Suppose the regular 7-gon can be constructed with straightedge and compass. Explain why $(\mathbb{Q}(\cos(2\pi/7) : \mathbb{Q}) = 2^k$ for some $k \in \mathbb{Z}^+$.

(c) If $F \equiv \mathbb{Q}(\cos(2\pi/7))$, show that $(F(i\sin(2\pi/7)) : F) = 1$ or 2.

(d) Use (b) and (c) to conclude that if the regular 7-gon can be constructed with straightedge and compass, then $(\mathbb{Q}(\cos(2\pi/7) + i\sin(2\pi/7)) : \mathbb{Q}) = 2^m$ for some $m \in \mathbb{Z}^+$.

(e) What can you conclude from (a) and (d)? What is your guess concerning the construction of the regular 11-gon, the regular 13-gon, the regular 23-gon, etc.?