MATH 113 – FINAL EXAM
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May 16, 2003

Time limit: 180 minutes

Name:

SID:

You may not consult any books or papers. You may not use a calculator or any other computing or graphing device other than your own head!

Unless instructed otherwise, you are required to justify all of your answers. An answer with no justification will receive little credit. Please write all answers in complete English sentences.

There are two extra blank pages at the end of the exam. You may use these for computations, but **I will not read them.** Please transfer all final answers to the page on which the question is posed.

> If you would like me to post your grade on the web, using your SID number instead of your name, please sign here:
>
> ....
>
> (If you do not have an SID, please see me after the exam.)

GOOD LUCK!

Problem:	Your score:	Total points
1		30 points
2		49 points
3		15 points
4		20 points
5		20 points
6		20 points
5		20 points
6		30 points
Total:		205 points

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CHEAT SHEET

- **The Chinese Remainder Theorem** $m, n \in \mathbb{N}$, $(m, n) = 1$. Then

$$x \equiv a \pmod{n} \text{ and } x \equiv b \pmod{m}$$

  has a solution, and any two solutions are congruent modulo mn.
- **Fundamental Homomorphism Theorem** Domain/ker $\cong$ Image
- **First Isomorphism Theorem** If $H \trianglelefteq G$ and $K \trianglelefteq$ with $K \subseteq H$, then $(G/K)/(H//K) \cong G/H$.
- **Second Isomorphism Theorem** If $N \trianglelefteq G$ and $H \leq G$, then $HN \leq G$, $H \cap N \trianglelefteq H$ and $(HN)/N \cong H/(H \cap N)$.
- If G acts on S, then
  - the orbit of a point $x \in S$ is $G \cdot x$, all the points that are G-multiples of x.
  - the stabilizer of a point $x \in S$ is G_x, all the elements of G sending x to itself.
- The centralizer of an element $x \in G$ is the same as the stabilizer of $x \in G$, when we think of G acting on itself by conjugation.
- The normalizer of a subgroup $H \leq G$ is the same as the stabilizer of $H \leq G$, when we think of G acting on its subgroups by conjugation.
- **Burnside's formula**
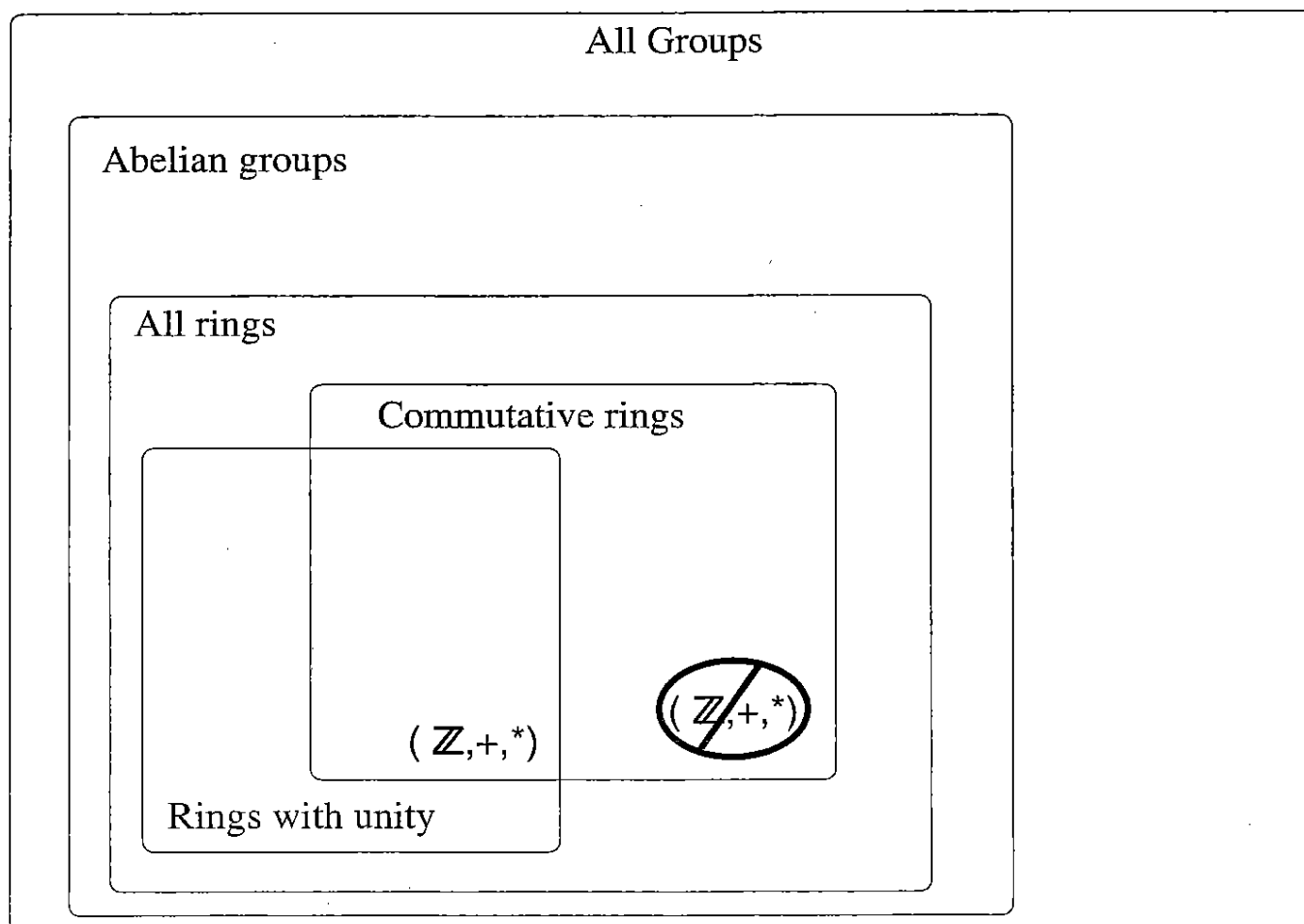
$$N = \frac{1}{|G|} \sum_{g \in G} |S^g|$$

- **Eisenstein's criterion** says that if $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ and there is a prime p dividing all of the coefficients except a_n, such that p^2 does not divide a_0, then $f(x)$ is irreducible over $\mathbb{Q}$
- A *commutative ring* is a ring with commutative multiplication
- A *ring with unity* is a ring with a multiplicative identity element
- An *integral domain* is a commutative ring with unity with no zero divisors
- A *principal ideal* is one generated by a single element of a ring
- A *principal ideal domain* is an integral domain all of whose ideals are principal
- A *prime ideal* is an ideal such that if a product is in the ideal, one of the two factors is in the ideal
- A *maximal ideal* is a proper ideal contained in no other proper ideal
- An element in an extension field is *algebraic* over the subfield if it is the root of a polynomial with coefficients in the subfield; otherwise, it's *transcendental*
- An element a is *idempotent* if $a^2 = a$

1. **(30 points - 3 points per example)** In this problem, you will provide several examples of groups and rings.
   - (a) Choose one region, and write **three** examples of that type of group or ring in the region.
   - (b) Choose two of the remaining five regions, and write **two** examples of that type of group or ring in each of those two regions.
   - (c) Finally, in the remaining three regions, write **one** example of that type of group or ring in those regions.

   Please be careful while placing your examples: an example must go in the most descriptive region possible. For example, I have placed the ring $(\mathbb{Z}, +, \cdot)$ in the region "commutative rings with unity," while I have crossed it out of "commutative rings." Moreover, I cannot place $(\mathbb{Z}, +)$ in the region for abelian groups, since there is a well-known second operation $\cdot$ on $\mathbb{Z}$ making it a ring.

   You **may not** use the example $(\mathbb{Z}, +, \cdot)$.

   All Groups

   Abelian groups

   All rings

   Commutative rings

   $(\mathbb{Z}, +, {}^*)$

   $(\cancel{\mathbb{Z}}, +, {}^*)$

   Rings with unity

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2. **(49 points) True/False.** Please determine whether the following statements are true or false, and circle your answer. Please justify your answer: give a proof or counterexample.

  (a) **(7 points)** There are exactly two equivalence relations on a set S that contains 4 elements.

True	False

  (b) **(7 points)** The element $4 + \sqrt[3]{2}$ is algebraic over $\mathbb{Q}$, with degree 3.

True	False

True/False. Please determine whether the following statements are true or false, and circle your answer. Please justify your answer: give a proof or counterexample.

(c) **(7 points)** The multiplicative cancellation law, "If $a \neq 0$ and $ab = ac$, then $b = c$," holds in any integral domain.

True	False

(d) **(7 points)** Let R be a commutative ring with unity, $R[x]$ the polynomials with coefficients in R, and $p(x) \in R[x]$ a polynomial of degree 1. Then $p(x)$ has a root in R.

True	False

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True/False. Please determine whether the following statements are true or false, and circle your answer. Please justify your answer: give a proof or counterexample.

(e) **(7 points)** Let $\mathbb{F}$ be a field, $\mathbb{F}[x]$ the polynomials with coefficients in $\mathbb{F}$, and $p(x) \in \mathbb{F}[x]$ a non-constant polynomial. The function $\psi : \mathbb{F}[x] \to \mathbb{F}[x]$ defined by $\psi(f(x)) = f(x) \cdot p(x)$ is a ring homomorphism.

True	False

(f) **(7 points)** Suppose that $\mathbb{F}$ is a field, and $\mathbb{D}$ and $\mathbb{E}$ are subfields of $\mathbb{F}$. Then $\mathbb{D} \cap \mathbb{E}$ is also a subfield of $\mathbb{F}$.

True	False

True/False. Please determine whether the following statements are true or false, and circle your answer. Please justify your answer: give a proof or counterexample.

(g) **(7 points)** The ring $\mathbb{Z}_{12}/\langle[4]_{12}\rangle$ is an integral domain.

True	False

3. **(15 points)** Show that a group G is abelian if and only if the function $\phi : G \to G$ defined by $\phi(g) = g^2$ is a homomorphism.

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4. **(20 points)** A jeweler has 15 different colors of beads, and he is going to create double-stranded bead necklaces. Suppose that you are asked the question, "How many distinct necklaces can he make with two strands of four beads each? An example necklace is shown below. We consider these necklaces up to rotation and
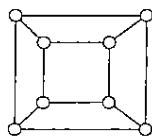
reflection. Note that the inside string is strictly different from the outside string. In the following figure, the two necklaces on the left are considered 'the same,' whilst the third necklace, on the right, is 'different' from the first two." You could answer
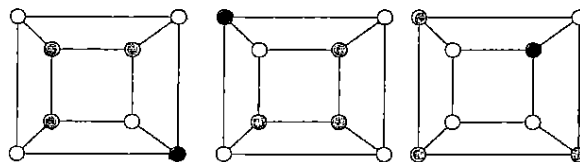
this question using Burnside's formula. Instead, please answer the following questions about your solution, in **complete English sentences.** (Use the back of this sheet, if necessary.)

(a) **(12 points)** Burnside's formula involves a group G acting on a set S. What is the set S in this case? What is the group G? What is the group action?

(b) **(8 points)** Using Burnside's formula, as stated on the cheat sheet (p. 2), you must compute terms $|S^g|$. In words, what is this term? Please compute it for one non-identity element $g \in G$.

5. **(21 points)** Suppose that G_1 and G_2 are groups, and $H_1 \trianglelefteq G_1$ and $H_2 \trianglelefteq G_2$ normal subgroups.

(a) **(7 points)** Prove that $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$.

(b) **(7 points)** Prove that $H_1 \times H_2 \trianglelefteq G_1 \times G_2$ is normal.

(c) **(7 points)** Prove that $(G_1 \times G_2)/(H_1 \times H_2) \cong (G_1/H_1) \times (G_2/H_2)$.

6. **(20 points)**

   (a) **(10 points)** Find all group homomorphisms from $\mathbb{Z}_6 \to \mathbb{Z}_{10}$.

   (b) **(10 points)** Which of the group homomorphisms in (a) are also ring homomorphisms?

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7. **(20 points)** Determine all the irreducible factors of the following polynomials over the field $\mathbb{Q}$.

   (a) **(10 points)** $f(x) = x^3 - 2x^2 + 3x + 5$

   (b) **(10 points)** $g(x) = x^4 - x^3 + 2x - 2$

8. **(30 points)** Let R be a commutative ring with unity, and let I and J be ideals in R. Please decide whether the following statements are true or false, and provide a proof or counterexample.

   (a) **(10 points)** If I contains a unit, then $I = R$.

True	False

   (b) **(10 points)** If I and J are both prime ideals, then $I \cap J$ is a prime ideal.

True	False

   (c) **(10 points)** If I and J are both principal ideals, then $I \cap J$ is a principal ideal.

True	False

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