Math 110, FINAL Exam, W.Arveson

Wednesday, 12/11/96, 12:30–3:30PM, 306 Latimer.

There are 100 points total. The point value of each problem is indicated.

1. *(10 points)*
Give an example of a 3×3 matrix A for which $A^2 \neq 0$ but $A^3 = 0$.

2. *(15 points)*
Let $(V, \langle \cdot\, , \cdot \rangle)$ be a real or complex inner product space and let $T \in \mathcal{L}(V)$.
(a) Define the adjoint operator T^*.
(b) Let M be an invariant subspace for M. Show that $M^\perp$ is invariant under T^*.

3. *(25 points)*
A real $n \times n$ matrix $A = (a_{ij})$ is called skew symmetric if

$$a_{ji} = -a_{ij}$$

for all $i, j = 1, 2, \ldots, n$. Fix such a matrix A, and consider it to be a linear operator acting by multiplication on column vectors in the inner product space $\mathbb{R}^n$, where

$$< x, y > = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

(a) Let $x \in \mathbb{R}^n$ and let $c \in \mathbb{R}$. Show that the norm of every vector of the form $x + cAx$ satisfies the inequality

$$\|x + cAx\| \geq \|x\|.$$

(b) Deduce that A has no (real) eigenvalues other than 0.
(c) Show that $U = (1 + A)(1 - A)^{-1}$ is well-defined, and satisfies the equation $U^*U = \mathbf{1}$.
(d) Deduce that U is an isometry: $\|Ux\| = \|x\|$ for every $x \in \mathbb{R}^n$.

4. *(10 points)*
Let T be a linear operator from $\mathbb{R}^2$ to $\mathbb{R}^4$ such that

$$\mathrm{ran} T = \{(x_1, x_2, x_3, x_4) : x_1 = x_3, \text{and } x_2 = x_4\}.$$

Show that T is one-to-one.

5. *(10 points)*
Let V be a complex vector space of dimension n and let $T \in \mathcal{L}(V)$ be an operator having only a single eigenvalue λ. Show that T must have the form

$$T = \lambda \mathbf{1} + S$$

where S is an operator satisfying $S^n = 0$.

6. *(10 points)*
Find a polynomial $f \in \mathcal{P}_3$ such that $f(0) = 0$, $f'(0) = 0$ and

$$\int_0^1 |2 + 3t - f(t)|^2 \, dt$$

is as small as possible.

7. *(20 points)*
Let V be a finite dimensional complex vector space, let $T \in \mathcal{L}(V)$ and let

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

with complex coefficients.
(a) Show that if $\lambda \in \mathbb{C}$ is an eigenvalue of T then $p(\lambda)$ is an eigenvalue of $p(T)$.
(b) Show that if μ is an eigenvalue of $p(T)$ then there is an eigenvalue λ of T such that $\mu = p(\lambda)$.