Math 110 - Final Exam

Spring 2000 – Nate Brown

1) (10pts) Assume $dim(V) = 6$, $U \subset V$ is a subspace and $dim(U) = 4$. Prove that there exist one dimensional subspaces $U_1,\ U_2 \subset V$ such that $V = U \oplus U_1 \oplus U_2$.

2) (10pts) Assume $T \in \mathcal{L}(V)$ is invertible and $\{v_1, \ldots, v_k\} \subset V$ is a linearly independent set of vectors. Prove that $\{T(v_1), \ldots, T(v_k)\}$ is also linearly independent.

3) (10pts) Let $< \cdot, \cdot >$ be the dot product on $\mathbb{C}^2$ (i.e. $< (x_1, y_1), (x_2, y_2) >= x_1\overline{x_2} + y_1\overline{y_2}$) and $T \in \mathcal{L}(\mathbb{C}^2)$ be defined by $T(x, y) = (ix + 2y, x + iy)$. Write down a formula for the adjoint of T with respect to $< \cdot, \cdot >$. (It does **not** suffice to just write down some matrix!)

4) (10pts) Assume $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{C}))$ has minimal polynomial $(z-1)(z+4)^2$. Find the matrix of T in Jordan form.

5) Let $U,\ V$ be vector spaces with $dim(U) > dim(V)$.

a) (10pts) Prove that if $T \in \mathcal{L}(U, V)$ then T is not injective.

b) (10pts) Construct some $T \in \mathcal{L}(U, V)$ which is surjective.

6) Define $T \in \mathcal{L}(M_2(\mathbb{C}))$ by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ia + 3c & -b + d \\ ic & 2d \end{pmatrix}$.

a) (10pts) Find all the eigenvalues of T. (Hint: Consider the matrix of T with respect to the canonical basis.)

b) (10pts) Compute the characteristic polynomial of T.

c) (10pts) What is the dimension of the generalized eigenspace corresponding to the eigenvalue i?

d) (10pts) Find $det(T)$.

7)a) (10pts) Construct an operator $T \in \mathcal{L}(M_2(\mathbb{C}))$ whose minimal polynomial is $z(z-3)^2$ and generalized eigenspace corresponding to 0 is one dimensional. (Give both a formula for the operator and it's matrix in Jordan form.)

b) (10pts) Construct an operator $S \in \mathcal{L}(M_2(\mathbb{C}))$ whose minimal polynomial is $z(z-3)^2$ and generalized eigenspace corresponding to 0 is two dimensional. (Give both a formula for the operator and it's matrix in Jordan form.)

8) For each $\lambda \in \mathbb{F}$ define $T_\lambda \in \mathcal{L}(\mathcal{P}_m(\mathbb{F}), \mathbb{F})$ by $T_\lambda(p) = p(\lambda)$.

a) (5pts) Prove that $ker(T_\lambda) = ker(T_{\tilde{\lambda}})$ if and only if $\lambda = \tilde{\lambda}$. (Hint: consider the polynomials $p_\gamma(z) = z - \gamma$.)

b) (10pts) Prove that $dim(ker(T_\lambda)) = m$.

c) (15pts) Let $U \subset \mathcal{P}_m(\mathbb{F})$ be the one dimensional subspace spanned by the polynomial $p(z) = z$. Prove that for every **nonzero** $\lambda \in \mathbb{F}$ there exists a subspace $U_\lambda \subset \mathcal{P}_m(\mathbb{F})$ such that *i)* $U_\lambda = U_{\tilde{\lambda}}$ if and only if $\lambda = \tilde{\lambda}$ and *ii)* $\mathcal{P}_m(\mathbb{F}) = U \oplus U_\lambda$ for each (nonzero) λ.

9) Let $B_1 = \{(1,0,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (0,0,0,1)\}$ be the canonical basis of $\mathbb{C}^4$, $B_2 = \{(1,0,0,0),\ (1,1,0,0),\ (0,0,1,0),\ (0,0,1,1)\}$ and $T \in \mathcal{L}(\mathbb{C}^4)$ be defined by $T(a,b,c,d) = ((1-i)a + (2i+1)b, -ia + (i+2)b, 2c, c)$.

a) (5pts) Compute $M(T, B_1, B_1)$.

b) (10pts) Compute the two change of basis matrices $M(I, B_1, B_2)$ and $M(I, B_2, B_1)$.

c) (5pts) Use part b) to show that $M(T, B_2, B_2) = \begin{pmatrix} 1 & i & 0 & 0 \\ -i & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

d) (10pts) Prove that T is **not** normal with respect the canonical dot product on $\mathbb{C}^4$.

e) (10pts) Prove that there exists a basis of $\mathbb{C}^4$ consisting of eigenvectors of T.

f) (10pts) Construct an inner product on $\mathbb{C}^4$ such that T is normal with respect to that inner product.