FINAL EXAM FOR MATHEMATICS 195 Spr 02

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INSTRUCTIONS: Please hand in your solutions to this test in my office, 907 Evans Hall, by 3:00 pm on Monday, May 20. You may not discuss this test with any person, nor consult any book or article. You may look at your own notes, the handouts distributed in class, and also the old set of lecture notes, downloaded from my website at http: //www.math.berkeley.edu/~evans/.

Please come by and talk with me if you have any questions about these problems. Good luck.

PROBLEM #1. Solve the SDE

$$\begin{cases} dX = \frac{1}{2}\sigma'(X)\sigma(X)dt + \sigma(X)dW \\ X(0) = 1 \end{cases}$$

where W is a one–dimensional Brownian motion and σ is a smooth, positive function.

(HINT: Let $f(x) = \int_1^x \frac{dy}{\sigma(y)}$ and set $g = f^{-1}$, the inverse function of f. Show $X := g(W)$.)

PROBLEM #2. Let $\mathbf{W}$ denote an n–dimensional Brownian motion, for $n \geq 3$. Write $\mathbf{X} = \mathbf{W} + x_0$, where the point x_0 lies in the region $U = \{1 < |x| < 3\}$

Calculate explicitly the probability that $\mathbf{X}$ will hit the outer sphere $\{|x| = 3\}$ before hitting the inner sphere $\{|x| = 1\}$.

(HINT: Check that the function

$$\Phi(x) = \frac{1}{|x|^{n-2}} = (x_1^2 + \cdots + x_n^2)^{\frac{2-n}{2}}$$

satisfies $\Delta\Phi = 0$ for $x \neq 0$. Modify Φ to build a function u which equals 0 on the inner sphere and 1 on the outer sphere, with $\Delta u = 0$ in U. Apply Itô calculus.)

PROBLEM #3. Let $\mathcal{F}$ be a filtration on a probability space with respect to a one-dimensional Brownian motion W. Assume that $G \in \mathbb{L}^2(0,T)$ is a step process, meaning that there exists a partition $P = \{0 = t_0 < t_1 < \cdots < t_m = T\}$ such that $G(t) \equiv G_k$ for $t_k \leq t < t_{k+1}$, $k = 0, \ldots, m-1$, and each G_k is an $\mathcal{F}(t_k)$-measurable random variable.

Prove that

$$I(t) = \int_0^t G\,dW \quad \text{is a martingale, for times } 0 \leq t \leq T.$$

(HINT: Show that if $0 \leq s < t \leq T$, then

$$E(I(t) - I(s) \mid \mathcal{F}(s)) = 0.)$$