MATH 121B: MIDTERM 2, SPRING, 2000

Total score: 100 points.

Problem 1.

(i) (10 points) *Directly from the definition of the* Γ *function, show that for* $p > 0$ *real,* $\Gamma(p+1) = p\Gamma(p)$.

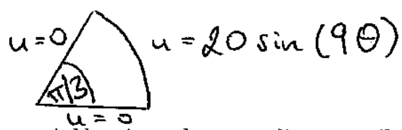
(ii) (10 points) *Find* $\int_0^1 x(\ln \frac{1}{x})^{3/2}\,dx$. *(Hint: change variables. You may use that* $\Gamma(1/2) = \sqrt{\pi}$.*)*

Problem 2. (25 points) *Using Frobenius' method of generalized (fractional) power series, solve the ODE*

$$9x^2y'' + 2(1+x^2)y = 0.$$

It suffices to find the first two non-zero terms of two linearly independent solutions of the ODE.

Problem 3. *Consider a sector of a circular plate of radius* $a = 2$ *and angle* $\pi/3$. *Suppose that its straight sides are kept at 0 temperature, and the curved side at temperature* $20\sin(9\theta)$. *Find the steady state temperature inside the sector as follows.*



(i) (17 points) *Separate variables in polar coordinates. Recall that, in two dimensions, the Laplacian in polar coordinates* (r, θ) *is given by*

$$\Delta_{\mathbb{R}^2} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Use this and the homogeneous boundary conditions to find the general form of the solution.

(ii) (8 points) *Find the coefficients of the separated solutions by making sure that the inhomogeneous boundary condition is satisfied.*

Problem 4. *Consider a cylinder of radius* 1, *height* 1. *Take the* z *axis as the axis of symmetry; and use cylindrical coordinates below. Suppose that the bottom,* $z = 0$, *and the side,* $r = 1$, *are kept at zero temperature, and the top,* $z = 1$, *at temperature* $100r^2\cos 2\theta$ *degrees. We want to find the steady state temperature in the cylinder.*

(i) (15 points) *Separate variables to find the general solution that satisfies all of the homogeneous boundary conditions. You may use that eigenfunctions* v *of the Laplacian on the unit disk satisfying Dirichlet boundary condition,* $v(1,\theta) = 0$, *are*

$$v = J_n(k_{nm}r)(A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)),$$

with $\Delta_{\mathbb{R}^2} v = -k_{nm}^2 v$; *here* k_{nm} *is the mth zero of* J_n.

(ii) (15 points) *Find the coefficients in the expansion, hence the solution of the original problem. Make sure that your final formula contains no integrals. You may use that the Bessel functions* J_p *satisfy* $\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$. *You may also use that* $\int_0^1 J_n(k_{nm}r)^2 r\,dr = J_{n+1}(k_{nm})^2/2$.