UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME134 Automatic Control Systems Fall 2004 (October 5th)

Midterm Exam I

Name: SOLUTION	**SID:**

Problem:	1	2	3	Total
Max. Grade:				100
Grade:				

Problem 1

Consider Van der Pol's equation for a nonlinear oscillator:

$$\ddot{I} - \mu(1 - I^2)\dot{I} + I = 0$$

This second order nonlinear model describes the current I in an electrical circuit with a nonlinear active element, such as a vacuum tube. It can be expressed as two first order ODEs in state space form:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \mu(1 - x_1^2)x_2 - x_1 \end{bmatrix} \tag{1}$$

where $x_1 = I$ and $x_2 = \dot{I}$.

(a) Find the equilibrium point x_e of Eq. (1).

(b) Find the linear approximation to Eq. (1) near x_e.

(c) Determine the stability of the linearized model about x_e, assuming $\mu > 0$ (is the system stable, limitedly stable, or unstable?).

(a) $\dot{x} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, $f_1 = x_2$
$f_2 = \mu(1 - x_1^2)x_2 - x_1$

$f_1 = 0 \longrightarrow x_{2e} = 0$

$f_2 = 0 \longrightarrow \mu(1 - x_{1e}^2)x_{2e} - x_{1e} = 0 \longrightarrow x_{1e} = 0$

$\Big\}\quad x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b)

$$
\left.
\begin{aligned}
&\frac{\partial f_1}{\partial x_1} = 0 \\
&\frac{\partial f_1}{\partial x_2} = 1 \\
&\frac{\partial f_2}{\partial x_1} = -2\mu x_1 x_2 - 1 \quad \longrightarrow \quad \left.\frac{\partial f_2}{\partial x_1}\right|_{x_e} = -1 \\
&\frac{\partial f_2}{\partial x_2} = \mu(1 - x_1^2) \quad \longrightarrow \quad \left.\frac{\partial f_2}{\partial x_2}\right|_{x_e} = \mu .
\end{aligned}
\right\}
A_{LIN} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}
$$

$$
\therefore \quad \underline{\dot{\xi} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix} \xi}
$$

(c)
$$
\begin{aligned}
\operatorname{eig}\{A_{LIN}\} &= \operatorname{roots}\left(\det \begin{bmatrix} s & -1 \\ 1 & s-\mu \end{bmatrix}\right) \\
&= \operatorname{roots}\left(s(s-\mu) + 1\right) \\
&= \operatorname{roots}\left(s^2 - \mu s + 1\right) \\
&= \frac{1}{2}\mu \pm \frac{1}{2}\sqrt{\mu^2 - 4}
\end{aligned}
$$

Since $\mu > 0$, there is always at least 1 eigenvalue with positive real part

$\therefore$ Unstable for all $\mu > 0$

Problem 2

Figure 1 shows a fluid/mechanical system. Power is transformed between the fluid and mechanical sides by an ideal piston, modelled with:

Power in = Power out

and

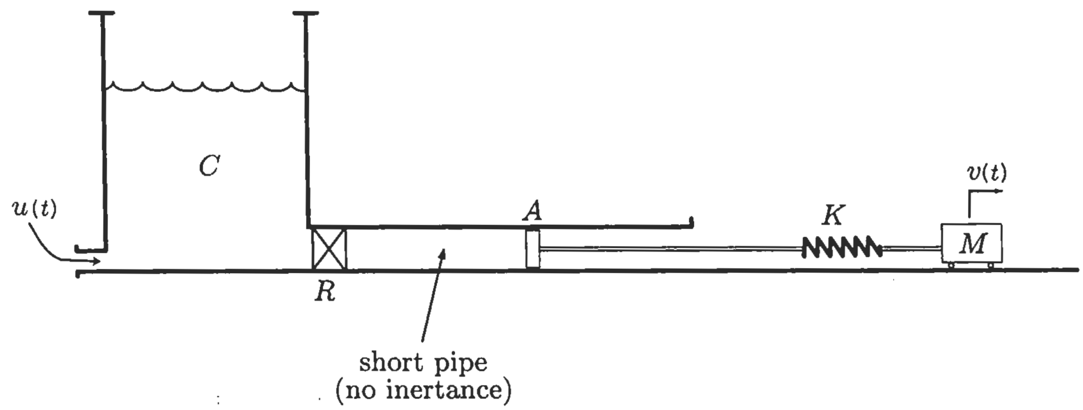
flow/A = piston speed

(a) Write the constitutive relation for each element.

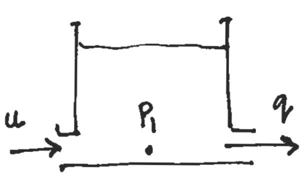
(b) Identify the independent storage elements. What is the order of the system?

(c) Put the model in state space form, with $v(t)$ as the output.

Figure 1:

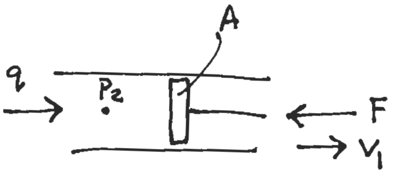


(a)

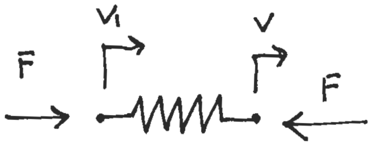


: $\dot{P}_1 = \frac{1}{C}(u - q)$

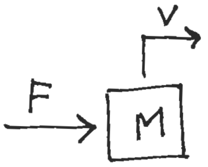
: $P_1 - P_2 = R \cdot q$



$$P_2 q = F V_1$$
$$q = A \cdot V_1$$



$$\dot{F} = K(V_1 - V)$$



$$\dot{V} = \frac{1}{M} F$$

(b) Storage elements: the tank, the spring, and the mass.

Order: 3

(c)
$$\begin{cases} \dot{P}_1 = \frac{1}{C}(u - q) \\ \dot{F} = K(V_1 - V) \\ \dot{V} = \frac{1}{M} F \end{cases}$$

Need to eliminate q, and V_1 using
$$\begin{cases} P_1 - P_2 = Rq & (\mathrm{I}) \\ P_2 q = F V_1 & (\mathrm{II}) \\ q = A V_1 & (\mathrm{III}) \end{cases}$$

$\mathrm{III} \to \mathrm{II}$: $P_2 (A V_1) = F V_1 \to P_2 = \frac{1}{A} F$

$\to \mathrm{I}$: $q = \frac{1}{R} P_1 - \frac{1}{R} P_2 = \frac{1}{R} P_1 - \frac{1}{AR} F$

$\to \mathrm{III}$: $V_1 = \frac{1}{A} q = \frac{1}{AR} P_1 - \frac{1}{A^2 R} F$

$$\begin{cases} \dot{P}_1 = \frac{1}{C}\left(u - \frac{1}{R} P_1 + \frac{1}{AR} F\right) \\ \dot{F} = K\left(\frac{1}{AR} P_1 - \frac{1}{A^2 R} F - V\right) \\ \dot{V} = \frac{1}{M} F \end{cases} \Rightarrow \boxed{\begin{bmatrix} \dot{P}_1 \\ \dot{F} \\ \dot{V} \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{CAR} & 0 \\ \frac{K}{AR} & \frac{-K}{A^2 R} & -K \\ 0 & \frac{1}{M} & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ F \\ V \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \\ 0 \end{bmatrix} u}$$

Problem 3

Given the following second order system:

$$\dot{x}(t) = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & c \end{bmatrix} x(t)$$

(a) Find the values of a, b, and c that make the transfer function from $u(t)$ to $y(t)$ equal to:

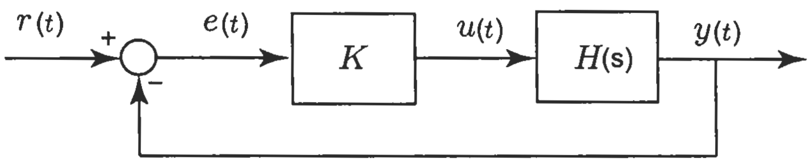
$$H(s) = \frac{10}{s^2 + 4s + 5}$$

(b) In Figure 2, find the transfer function from $r(t)$ to $y(t)$.

(c) Assuming $r(t)$ is a unit step input, and $K = 1$, find $\lim_{t\to\infty} y(t)$.

(d) Find the value of K such that the closed-loop damping ratio is $\xi = 0.5$.

Figure 2:



(a) $$H(s) = \begin{bmatrix} 0 & c \end{bmatrix} \begin{bmatrix} s-a & -1 \\ -b & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} 0 & c \end{bmatrix} \begin{bmatrix} s & 1 \\ b & s-a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{s(s-a) - b}$$

$$= \frac{cb}{s^2 - as - b}$$

$\therefore$ $cb = 10$

$-a = 4$

$-b = 5$

$\Longrightarrow$ $\underline{a = -4, \quad b = -5, \quad c = -2}$

(b) $E = R - Y$

$U = KE$

$Y = HU = HKE = HK(R-Y)$

$\therefore (1+HK)Y = HKR$

$$\therefore Y = \frac{HK}{1+HK} R$$

$$= \frac{\frac{10K}{s^2+4s+5}}{1 + \frac{10K}{s^2+4s+5}} R$$

$$= \underbrace{\frac{10K}{s^2+4s+5+10K}}_{\text{T.F. from } r \to y.} R$$

(c) $R = \frac{1}{s}$.

Is it stable? Check roots of $s^2 + 4s + 15$

$$p = -\frac{4}{2} \pm \frac{1}{2}\sqrt{16 - 4 \cdot 15} = -2 \pm \sqrt{11}\, j \quad ✓ \text{ YES.}$$

∴ We can use the final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s \cdot Y(s) = \lim_{s\to 0} \cancel{s} \cdot \frac{10}{s^2+4s+15} \cdot \frac{1}{\cancel{s}} = \frac{10}{15} = \frac{2}{3}$$

(d) $\begin{cases} \omega_n^2 = 5 + 10K \\ 2\zeta\omega_n = 4 \end{cases} \longrightarrow 4\zeta^2\omega_n^2 = 16$

$\therefore \zeta^2(5+10K) = 4$

$\therefore 5 + 10K = 4/\zeta^2 = 16 \longrightarrow K = \frac{16-5}{10} = \frac{11}{10}$