MIDTERM
MATH 121A
SPRING 2000

[1] 10 points

Solve

$$y''(t) + y(t) = \sin(t) \qquad ; \qquad y(0) = 0,\ y'(0) = 0.$$

[2] 5 points

Find the Laplace transform of

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ 3e^{-(t-2)} & \text{if } 2 \le t \end{cases} .$$

[3] **5 × 2 points**

Let $f(x)$ and $g(x)$ be absolutely integrable functions on $\mathbb{R}$. Let $\mathcal{F}[f]$ denote the Fourier transform of $f(x)$.

(i) What formula defines the Fourier transform of $f(x)$?

(ii) What formula defines $f \star g$?

(iii) Express $\mathcal{F}[f'(x)]$ in terms of $\mathcal{F}[f]$.

(iv) Express $\mathcal{F}[f \star g]$ in terms of $\mathcal{F}[f]$ and $\mathcal{F}[g]$.

(v) Express f in terms of $\mathcal{F}[f]$.

[4] 5 points

Evaluate

$$\oint_C \frac{dz}{z},$$

where C is the circle of radius 2 centered at 0, *oricntcd counterclockwise.*

[5] **10 points**

Let $\mathbf{z} = \sqrt{3} - \mathbf{i}$.

(i) Write z as $re^{i\theta}$, i.e., find r and θ.

(ii) What is z^3?

(iii) What are $\Re(1/z)$ and $\Im(1/z)$, the real and imaginary parts of $1/z$?

(iv) What is $\Im(\overline{i\overline{z}})$?

4

[6] 5 points

Find the radius of convergence of $\sum_{n=1}^{\infty} 2nz^{2n-1}$.

[7] 5 points

What does $\sum_{n=1}^{\infty} 2nz^{2n-1}$ equal where it converges?

[8] **10 points**

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be analytic, and let $u(x, y) = \Re(f(x + iy))$.

Show that

$$\frac{\partial^2}{\partial x^2} u(x, y) \ + \ \frac{\partial^2}{\partial y^2} u(x, y) \ = \ 0.$$

[Hint: Use the Cauchy-Riemann equations]

[9] Statement of definitions 2×5 points

(i) Let S be a set of real numbers. *Carefully* define $\sup S$, the supremum (or least upper bound) of S.

(ii) Let $f : D \longrightarrow \mathbb{C}$ be a complex-valued function on a domain D, and for each $n \in \mathbb{N}$, let $f_n : D \longrightarrow \mathbb{C}$. By definition, $\{f_n\}$ **converges uniformly** to f (on D) if and only if . . .

[10] **10 points**

Solve Schrödinger's equation on $\mathbb{R}$ using the Fourier Integral:

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(x,t) &= -\frac{\partial^2}{\partial x^2}\psi(x,t) \\ \psi(x,0) &= f(x). \end{aligned}$$

[11] Proof problem 20 points

Let $f : D \longrightarrow \mathbb{C}$ be defined on an open domain D of the complex plane, and let w be a point in D. Recall that

$$\lim_{z \to w} f(z) = c,$$

if and only if the following condition is met: For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - c| < \epsilon$ if $|z - w| < \delta$.

Prove one direction — your choice, either "if" or "only if" — of the following theorem. For extra credit, prove both directions.

Theorem:

$$\lim_{z \to w} f(z) = c$$

if and only if $\{f(z_n)\}$ converges to c whenever $\{z_n\}$ converges to w.