Second Midterm Examination
Friday April 3 2009
Closed Books and Closed Notes

Question 1 *A Linkage System*

As shown in Figure 1, a mechanical linkage consists of a system of 4 particles which are connected by a set of identical massless rods each of length L to a central point C. The masses of the particles are identical $m_1 = m_2 = m_3 = m_4$. The particle of mass m_3 is connected to a fixed point O by a linear spring of stiffness K and unstretched length L_0. The mechanism moves on a smooth horizontal plane. A gravitational force acts normal to this plane.

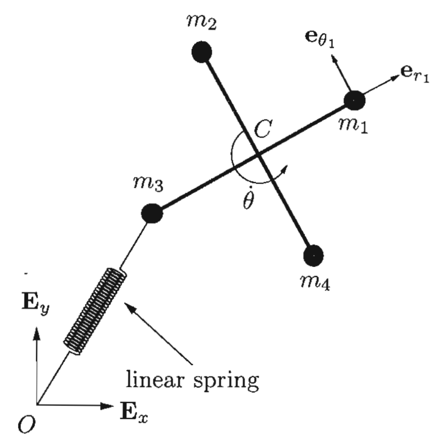


Figure 1: A system of particles moving on a smooth horizontal plane.

(a) *(6 Points)* Starting from the representations for the position vector of the center of mass C and the position vector of the particle of mass m_1,

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y, \qquad \mathbf{r}_1 = L\mathbf{e}_{r_1} + \mathbf{r}, \tag{1}$$

establish expressions for the linear momentum $\mathbf{G}_1$ and angular momenta relative to O and C of the particle of mass m_1.

(b) *(4 Points)* For the system of 4 particles,

$$\mathbf{G} = m\left(\dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y\right), \qquad \mathbf{H}_C = mL^2\dot{\theta}\mathbf{E}_z, \tag{2}$$

where $m = m_1 + m_2 + m_3 + m_4$ and $\dot{\theta} = \dot{\theta}_i$ $(i = 1, 2, 3, 4)$. Using (2), what is $\mathbf{H}_O$?

(c) *(3 Points)* Draw a free-body diagram of the system of particles. In your solution, give a clear expression for the spring force.

(d) *(2 Points)* Using a balance of linear momentum for each particle show that the normal force acting on each particle is equal and opposite to the gravitational force.

(e) *(5 Points)* Show that $\mathbf{H}_O$ is conserved. In addition, show that $\mathbf{H}_C$ is not conserved.

(f) *(5 Points)* Starting from the work-energy theorem for a system of particles, show that the total energy E of the system of particles is conserved.

Question 2

A System of Two Particles

25 Points

As shown in Figure 2, a particle of mass m_1 is at rest and is attached to a smooth horizontal surface by two identical linear springs of stiffnesses K and unstretched lengths L_0. At time t = 0, a particle of mass m_2 traveling with a velocity vector $v_0\mathbf{E}_x$ impacts the particle of mass m_1. After the collision both particles adhere to each other, and can be considered as a particle of mass $m_1 + m_2$.

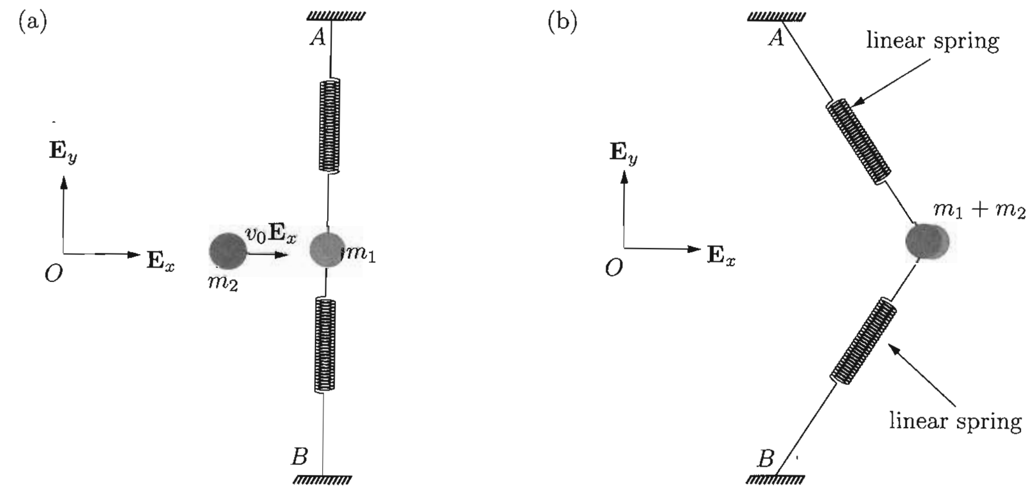


Figure 2: A system of two particles: (a) Prior to impact at $t = 0$, and (b) following the impact. The position vectors of the fixed points A and B are $\mathbf{r}_A = H\mathbf{E}_y + W\mathbf{E}_x$ and $\mathbf{r}_B = -H\mathbf{E}_y + W\mathbf{E}_x$, respectively

(a) *(4 Points)* Starting from the representation

$$\mathbf{r}_1 = (x + W)\,\mathbf{E}_x, \tag{3}$$

where W is a constant, establish representations for the linear momentum, kinetic energy, and acceleration of the particle of mass $m_1 + m_2$ after the collision.

(b) *(5 Points)* Show that the velocity of the particle of mass $m_1 + m_2$ immediately following the collision is

$$\dot{x}(t = 0) = \frac{m_2}{m_1 + m_2} v_0. \tag{4}$$

Verify that the kinetic energy of the system is not conserved during the collision.

(c) *(6 Points)* Draw a freebody diagram of the particle of mass $m_1 + m_2$ following the collision. Give clear expressions for the spring forces acting on the particle.

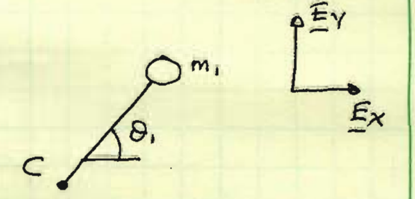
(d) *(5 Points)* Consider the system after impact. Starting from $\dot{T} = \mathbf{F} \cdot \mathbf{v}$ for a single particle, show that the total energy E of the particle of mass $m_1 + m_2$ is conserved. In your solution, give an expression for E.

(e) *(5 Points)* Following the impact of the particle of mass m_2, if $H = L_0$, show that the maximum displacement x_{max} of the particle of mass $m_1 + m_2$ is given by

$$x^2_{\text{max}} = \left(\left(\sqrt{\frac{m_2}{2K} \left(\frac{m_2}{m_1 + m_2} \right)} \right) v_0 + H \right)^2 - H^2. \tag{5}$$

$\boxed{\text{Problem 1}}$

(a)



$$\mathbf{r}_1 = L\mathbf{e}_{r_1} + \mathbf{r}$$

$$\mathbf{v}_1 = L\dot{\theta}_1\mathbf{e}_{\theta_1} + \mathbf{v}$$

$$\mathbf{e}_{r_i} = \cos\theta_i\,\mathbf{E}_x + \sin\theta_i\,\mathbf{E}_y$$

$$\dot{\theta}_i = \dot{\theta}$$

$$\mathbf{G}_1 = m_1\mathbf{v}_1 \quad ; \quad \mathbf{H}_{c_1} = L\mathbf{e}_{r_1} \times (m_1 L\dot{\theta}_1\mathbf{e}_{\theta_1} + m_1\mathbf{v})$$

$$= m_1 L^2\dot{\theta}\,\mathbf{E}_z + m_1 L(\dot{y}\cos\theta_1 - \dot{x}\sin\theta_1)\,\mathbf{E}_z$$

$$\mathbf{H}_{o_1} = \mathbf{H}_{c_1} + \mathbf{r} \times (m_1 L\dot{\theta}_1\mathbf{e}_{\theta_1} + m_1\mathbf{v})$$

(b) $\mathbf{G} = (m_1 + m_2 + m_3 + m_4)\mathbf{v}$

$$\mathbf{H}_c = \sum_{i=1}^{4}\mathbf{H}_{c_i} = (m_1 + m_2 + m_3 + m_4)L^2\dot{\theta}\,\mathbf{E}_z + m_1 L\sum_{i=1}^{4}(\dot{y}\cos\theta_i - \dot{x}\sin\theta_i)\,\mathbf{E}_z$$

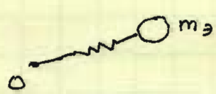
$$= mL^2\dot{\theta}\,\mathbf{E}_z + \mathbf{0} \qquad \text{where} \qquad m = m_1 + m_2 + m_3 + m_4$$

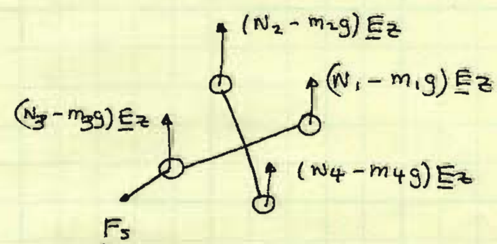
$$\mathbf{H}_o = \mathbf{H}_c + \mathbf{r} \times m\mathbf{v}$$

$$= \mathbf{H}_c + (x\mathbf{E}_x + y\mathbf{E}_y) \times m(\dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y)$$

$$= mL^2\dot{\theta}\,\mathbf{E}_z + m(x\dot{y} - y\dot{x})\,\mathbf{E}_z$$

(c)





$$\mathbf{F}_s = -K\left(\|\mathbf{r}_3 - \mathbf{0}\| - L_0\right)\frac{\mathbf{r}_3}{\|\mathbf{r}_3\|}$$

(d) $\mathbf{F}_i = m_i\mathbf{a}_i \Rightarrow$ as $\mathbf{a}_i \cdot \mathbf{E}_z = 0 \Rightarrow N_i - m_i g = 0 \Rightarrow N_i = m_i g.$

(e)$^+$ $\dot{\mathbf{H}}_o = \sum_{i=1}^{4}\mathbf{r}_i \times \mathbf{F}_i = \sum_{i=1}^{4}\mathbf{r}_i \times (N_i - m_i g)\mathbf{E}_z + \mathbf{r}_3 \times \mathbf{F}_s = \mathbf{0} + \mathbf{0}$

$\Rightarrow$ $\mathbf{H}_o$ is conserved

$\dot{\mathbf{H}}_c = \dot{\mathbf{H}}_o - \mathbf{r} \times \mathbf{F} = \mathbf{0} - \mathbf{r} \times \mathbf{F}_s \neq \mathbf{0} \Rightarrow$ $\mathbf{H}_c$ is not conserved.

(f)$^+$ $\dot{E} = \sum_{i=1}^{4}\mathbf{F}_{nc_i} \cdot \mathbf{v}_i = \sum_{i=1}^{4} N_i\mathbf{E}_z \cdot \mathbf{v}_i = 0$ as $\mathbf{v}_i \perp \mathbf{E}_z$

$+$ Sufficient to ignore tension forces in rods here.

Problem 2

a) After the collision $\quad \underline{r} = \underline{r}_1 = \underline{r}_2$

Hence $\quad \underline{G} = (m_1+m_2)\dot{\underline{r}} = (m_1+m_2)\dot{x}\,\underline{E}_x$

$$T = \tfrac{1}{2} m_1 \underline{v}_1\cdot\underline{v}_1 + \tfrac{1}{2} m_2 \underline{v}_2\cdot\underline{v}_2 = \tfrac{1}{2}(m_1+m_2)\dot{x}^2$$

$$\underline{a} = \ddot{\underline{r}} = \ddot{x}\,\underline{E}_x$$

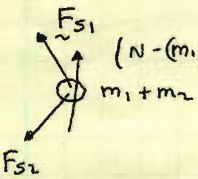
b) Linear momentum is conserved during the collision

$$(m_1+m_2)\dot{x}(t_0) = m_2 v_0 \quad \Rightarrow \quad \dot{x}(t=t_0=0) = \frac{m_2 v_0}{m_1+m_2}$$

$$T_{before} = \tfrac{1}{2} m_1 v_0^2 \qquad T_{after} = \tfrac{1}{2}(m_1+m_2)\frac{(m_2 v_0)^2}{(m_1+m_2)^2} = \frac{1}{2}\frac{m_2^2}{m_1+m_2} v_0^2$$

$$T_{after} - T_{before} = \tfrac{1}{2} m_2 v_0^2 \left(\frac{m_2}{m_1+m_2} - 1\right) < 0 \quad \Rightarrow \text{ Energy is lost}$$

c)



$$\underline{F}_{s1} = -K\left(\|\underline{r}-\underline{r}_A\| - L_0\right)\frac{\underline{r}-\underline{r}_A}{\|\underline{r}-\underline{r}_A\|}$$

$$\underline{F}_{s2} = -K\left(\|\underline{r}-\underline{r}_B\| - L_0\right)\frac{\underline{r}-\underline{r}_B}{\|\underline{r}-\underline{r}_B\|}$$

d) $$\dot{T} = \underline{F}\cdot\underline{v} = \underline{F}_{s1}\cdot\underline{v} + \underline{F}_{s2}\cdot\underline{v} + (N-(m_1+m_2)g)\underline{E}_z\cdot\underline{v}$$

$$= -\frac{d}{dt}\left(U_{s1}+U_{s2}\right) + 0$$

where $\quad U_{s1}+U_{s2} = \tfrac{1}{2}K\left(\|\underline{r}-\underline{r}_A\| - L_0\right)^2 + \tfrac{1}{2}K\left(\|\underline{r}-\underline{r}_B\| - L_0\right)^2$

Hence $\quad \dfrac{d}{dt}\left(E = T + U_{s1} + U_{s2}\right) = 0 \quad \Rightarrow E$ is conserved.

e) As Energy is conserved $\quad E = E_0 = \tfrac{1}{2}K(H-L_0)^2 + \tfrac{1}{2}K(H-L_0)^2 + T_{after}$

Hence $$E = K(H-L_0)^2 + \frac{1}{2}\frac{m_2^2}{m_1+m_2}v_0^2$$

Now max displacement occurs when $\underline{v} = \underline{0}$: $\quad \underline{r}-\underline{r}_A = x^*\underline{E}_x + H\underline{E}_y$, $\quad x^* = x_{max}$

$\quad \underline{r}-\underline{r}_B = x^*\underline{E}_x - H\underline{E}_y$

$$\Rightarrow K(H-L_0)^2 + \frac{1}{2}\frac{m_2^2}{m_1+m_2}v_0^2 = K\left(\sqrt{x_{max}^2+H^2} - L_0\right)^2$$

$$\Rightarrow \text{As } H = L_0 : \quad x_{max}^2 = \left(\sqrt{\left(\frac{m_2}{2K}\right)\left(\frac{m_2}{m_1+m_2}\right)v_0^2} + L_0\right)^2 - H^2$$