Final Exam—100 points

1 *(4 points each)*. In each part, determine whether or not the given series or integral converges, and find its value if it does:

a. $.9 - .99 + .999 - .9999 + .99999 - .999999 + \cdots$

b. $\displaystyle\int_{e^2}^{+\infty} \frac{dx}{x(\ln x)^{1.5}}$

c. $\displaystyle\frac{1}{2^2} + \frac{2}{3^2} + \frac{3}{4^2} + \cdots.$

2a *(4 points)*. Find the area of the parallelogram spanned by the vectors $\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. [We drew a picture of this on the board during the exam, because some students didn't understand what "spanned meant."

2b *(5 points)*. Find the area of the plane region bounded by the three polar curves $r = \sec\theta$, $\theta = 0$, and $\theta = \pi/4$. Sketch this region.

3 *(4 points each)*. In each part, determine whether or not the given limit exists or the given series converges. Find its value if it does:

a. $\displaystyle\sum_{n=1}^{\infty} \left[\sin\left(\frac{n+1}{n}\right) - \sin\left(\frac{n+2}{n+1}\right)\right].$

b. $\displaystyle\lim_{t \to 0+} t^{1/\ln t}.$

c. $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} 3^n$, where $f(x) = \dfrac{x}{1-x}$.

4 *(5 points each)*.

a. Find $\displaystyle\lim_{x \to \pi/2} \frac{2x - \pi}{\tan^{-1} x}$. [In the exam as distributed, the numerator and denominator of the fraction were interchanged; we corrected the problem to the given one by writing the desired fraction on the board.]

b. Use the chain rule for partial derivatives to calculate $\dfrac{d}{dt}(x^y)$ when $x = t$, $y = \dfrac{1}{\ln t}$.

5a *(4 points)*. Express in terms of definite integrals the y-coordinate ("$\bar{y}$") of the centroid of the cycloid

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases} \qquad 0 \le t \le 2\pi.$$

[There was an accompanying Mathematica-generated illustration of the cycloid. During the exam, we explained that we were interested in the cycloid itself as a curve, not in any two-dimensional figure.]

5b *(7 points)*. Calculate $\int \frac{4x^2+9x+3}{x^3+2x^2+x}\,dx$.

6a *(4 points)*. A function $F(x,y)$ satisfies $\frac{\partial F}{\partial x}(3,4) = 2$, $\frac{\partial F}{\partial y}(3,4) = -1$ What is the largest possible directional derivative $\frac{dF}{ds}$ computed at the point $(3,4)$?

6b *(5 points)*. A function $G(x,y)$ satisfies $\frac{\partial^2 G}{\partial x^2}(1,0) = 2$, $\frac{\partial^2 G}{\partial y^2}(1,0) = -1$. Which statements are compatible with these data (check ✔ all that apply):

- ☐ The point $(1,0)$ is not a critical point for G.
- ☐ The point $(1,0)$ is a local maximum for G.
- ☐ The point $(1,0)$ is a local minimum for G.
- ☐ The point $(1,0)$ is a saddle point for G.

6c *(4 points)*. Given that $\mathbf{grad}\, H(2,0) = 3\mathbf{i}+\mathbf{j}$, calculate $\lim_{T\to 0} \frac{H(2+\frac{T}{\sqrt{2}}, \frac{T}{\sqrt{2}}) - H(2,0)}{T}$.

7a *(6 points)*. Find a number a such that the two lines $\frac{x-1}{3} = \frac{y-a}{2} = \frac{z-4}{-1}$ and $\frac{x-1}{1} = \frac{y-5}{-2} = \frac{z-2}{-1}$ intersect. At what point do they intersect?

7b *(4 points)*. Do all the different lines (for varying a) $\frac{x-1}{3} = \frac{y-a}{2} = \frac{z-4}{-1}$ lie in a single plane? If so, find an equation for the plane.

8a *(7 points)*. For what values of x does the series $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} 3^{3n} x^n$ diverge? For what values of x does it converge conditionally? For what values does it converge absolutely?

8b *(5 points)*. At what point on the graph of $y = e^x$ is the graph most curved? $[k = \frac{y''}{(1+y'^2)^{3/2}}]$

9a *(5 points)*. Obtain the Taylor series of $1 - 6\sin^2 x$ about $x = 0$. [Hint: $\cos 2x = \cos^2 x - \sin^2 x$.]

9b *(6 points)*. Find the point on the plane $3x + 4y - 5z = 200$ that is closest to the origin.