Department of Mathematics, University of California, Berkeley

Math 1B

Alan Weinstein, Spring, 2002

Final Examination, Tuesday, May 21, 2002

Instructions. Be sure to write on the front cover of your blue book: (1) your name, (2) your Student ID Number, (3) your TA's name (Tathagata Basak, Tameka Carter, Alex Diesl, Clifton Ealy, Peter Gerdes, John Goodrick, Matt Harvey, George Kirkup, Andreas Liu, Rob Myers, or Kei Nakamura).

Read the problems very carefully to be sure that you understand the statements. Show all your work as clearly as possible, and circle each final answer to each problem. **When doing a computation, don't put an "=" sign between things which are not equal. When giving explanations, write complete sentences. You are more likely to get credit for a solution if it is written clearly.**

1. [10 points] The error bound for the trapezoidal rule applied to the integral of $f(x)$ on an interval of length L, with n subintervals, is:

$$|E_T| \leq \frac{M_2 L^3}{12n^2},$$

where M_2 is any estimate for the second derivative of f, i.e. $|f''(x)| \leq M_2$ for all x in the interval of integration.
Suppose that we want to integrate the function $x^{100} + 2$ over the interval from 0 to 1, with an error of at most 0.01. Find a real number N such that using n subintervals, where n is an integer greater than N, will guarantee the required accuracy.

2. [10 points] Write each of the following complex numbers in the form $a + bi$.
(A) $1/(6i + 4)$
(B) $(1 - i)^{10}$
(C) e^{2+3i}

3. [10 points] A particle moving along a straight line has velocity $v(t) = 3t^2e^{-t}$ meters per second after t seconds.
(A) How far does it move during the first 10 seconds?
(B) How far does it move during the next 10 seconds?
(C) Where is the closest place to the origin that you could put a wall which will never be reached by the particle, no matter how long it moves? (Assume that the particle starts at the origin and moves toward the well.)

4. [10 points] For which values of the real number q is the improper integral $\int_1^\infty x^q \sin^2(1/x)\, dx$ convergent? (You must justify your answer.)

5. [10 points] Find the average value of the function $f(x) = 4 + \sin^2 3x$ on the interval $[0, 2\pi]$.

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6. [10 points] The half life of radium-226 is 1590 years.
(A) Let $m(t)$ be the mass at time t of the radium-226 in a sample. Write down a first-order differential equation which describes how $m(t)$ changes with time.
(B) How long does it take for the mass of the radium-226 in the sample to drop to $1/e$ times its initial value?

7. [10 points] For which (integer) value(s) of the exponent k does the sequence

$$a_n = \frac{2n^k + 3}{3n^k + 2}$$

converge? Find the limit when it exists. (The answer may depend on k.)

8. [10 points] Find two different functions which are solutions of the differential equation $4y'' + 5y' + y = e^x$. There should be no arbitrary constants in your answers.

9. [10 points] Find the first four terms of the power series solution of the initial value problem

$$d^2y/dx^2 + dy/dx + x^2y = 3y \qquad y(0) = 2, y'(0) = 1.$$

10. [10 points] Find the sum of the series $\sum_{n=3}^{\infty} \frac{1}{n(n-1)}$. [Hint: Expand the general term of the series in partial fractions.]

11. [10 points] Determine whether each of the following series converges or diverges. You must justify your answer.
(A)

$$\sum_{n=1}^{\infty} \left(\frac{3n+2}{2n+3}\right)^n$$

(B)

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

(C)

$$\sum_{n=1}^{\infty} (-1)^n \ln(1 + \frac{1}{n})$$

12. [10 points] Tell whether the following statement is true or false, and explain your reasoning: If a solution of the differential equation $dy/dt = (y-1)(y-2)(y-3)(y-5)(y-8)$ has the value $y = 4$ when $t = 0$, then, for this solution, $y > 4$ when $t = 17$. [Note: your explanation may include a figure.]