P. Vojta

Math 1BM Final Exam

Sat 20 May 2000

Some Formulas

1. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
2. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
3. $\sin A \cos B = \frac{1}{2}\left[\sin(A - B) + \sin(A + B)\right]$
4. $\sin A \sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$
5. $\cos A \cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$
6. $\displaystyle\int \tan u\, du = \ln|\sec u| + C$
7. $\displaystyle\int \sec u\, du = \ln|\sec u + \tan u| + C$
8. $\displaystyle\int \tan^n u\, du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\, du$
9. $\displaystyle\int \sec^n u\, du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\, du$
10. Weierstrass substitution: $t = \tan\left(\frac{x}{2}\right)$; $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2}dt$.
11. Binomial series:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \qquad \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \quad (n \ge 1); \qquad \binom{k}{0} = 1$$

12. $\ln(1 - x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dots.$
13. $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dots.$

1. (35 points) Find:
   (a). $\int_0^2 \sqrt{x^2+4}\,dx$
   (b). $\int_1^e (\ln x)^2 dx$
   (c). $\int_0^{1/2} \frac{\arctan x}{x}\,dx$

2. (12 points) If the curve $y = \sqrt{2x - x^2}$, $0 \le x \le 1$, is rotated about the x-axis, find the area of the resulting surface.

3. (14 points) Describe how one can compute $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ to within 0.01.
   (You do not need to actually carry out the computation, but if your answer involves, say, the n^{th} partial sum, then you should say what n is.)

4. (25 points) Determine whether the following series converge absolutely, converge conditionally, or diverge:
   (a). $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n+1}}$
   (b). $\frac{2}{1} - \frac{1}{2} - \frac{1}{3} + \frac{2}{4} - \frac{1}{5} - \frac{1}{6} + \frac{2}{7} - \dots$

5. (25 points) Determine whether the following series converge absolutely, converge conditionally, or diverge:
   (a). $\sum_{n=1}^{\infty} \left(\arctan\left(7 + \frac{1}{n}\right) - \arctan 7\right)$
   (b). $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

6. (14 points) Find the Maclauren series for $\frac{1}{\sqrt{1-x^2}}$.

7. (20 points) Solve the differential equation
$$y' = \frac{\ln x}{xy + xy^3}\ .$$

8. (20 points) Solve the boundary-value problem $y'' + 2y' + 2y = 0$, $y(0) = 0$, $y(\pi/2) = 1$.

9. (24 points) Find the general solution of the differential equation $y'' + 2y' + y = \frac{e^{-x}}{x}$.

10. (24 points) Find the (series) solution of the initial-value problem
$$y'' - xy' - y = 0 \qquad y(0) = 0\ , \qquad y'(0) = 1\ .$$

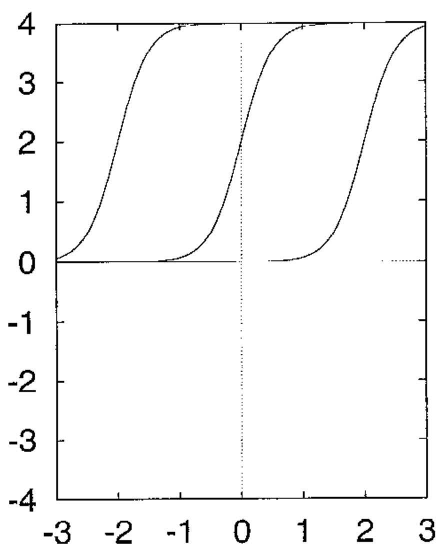
11. (12 points) Match thc equation to the graph. You may assume that each graph belongs to one equation.

____ $y' = x^2 + y$ ____ $y' = x^3 + xy$

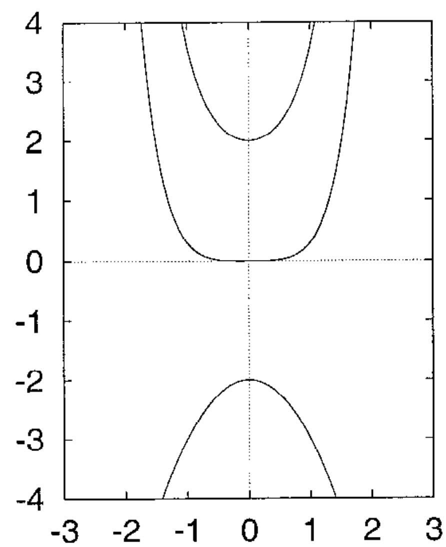
____ $y' = y(4 - y)$ ____ $y' = x^2 - y$

Suggestion: Do not solve the equations.

A.



B.



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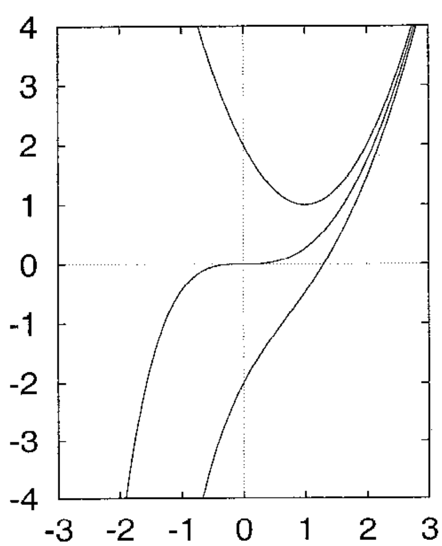

D.