UCB Math 1B, Fall 2009: Final Exam

Prof. Persson, December 18, 2009

Name: Solutions

SID: ____________

Section: Circle your discussion section below:

Sec	Time	Room	GSI
01	MW 8am - 9am	75 Evans	G. Melvin
02	MW 8am - 9am	5 Evans	T. Wilson
03	MW 10am - 11am	75 Evans	D. Cristofaro-Gardiner
04	MW 10am - 11am	3113 Etcheverry	E. Kim
05	MW 11am - 12pm	81 Evans	G. Melvin
06	MW 12pm - 1pm	5 Evans	T. Wilson
07	MW 1pm - 2pm	2 Evans	A. Tilley
09	MW 2pm - 3pm	247 Dwinelle	D. Cristofaro-Gardiner
10	MW 3pm - 4pm	4 Evans	E. Kim
11	MW 4pm - 5pm	3113 Etcheverry	A. Tilley
12	TT 11:30am - 2pm	230C Stephens	L. Martirosyan

Other/none, explain: ____________

Grading

1	/ 10
2	/ 10
3	/ 10
4	/ 10
5	/ 15
6	/ 15
7	/ 15
8	/ 15
	/100

Instructions:

- One double-sided sheet of notes, no books, no calculators.
- Exam time 180 minutes, do all of the problems.
- You must justify your answers for full credit.
- Write your answers in the space below each problem.
- If you need more space, use reverse side or scratch pages. Indicate clearly where to find your answers.

1. (10 points) Solve the initial value problem $y' + y\cos x = y^2 \cos x$, $y(0) = 2$.

Seperable! $y' = \cos x \cdot (y^2 - y)$

$$\frac{dy}{y^2 - y} = \cos x\, dx$$

$$\int \frac{1}{y(y-1)}\, dy = \int \cos x\, dx$$

$$\int \frac{1}{y-1} - \frac{1}{y}\, dy = \sin x + c$$

$$\ln\left|\frac{y-1}{y}\right| = \sin x + c$$

$$\left|\frac{y-1}{y}\right| = e^{\sin x} e^{c}$$

$$1 - \frac{1}{y} = Ae^{\sin x}$$

$$y = \frac{1}{1 - Ae^{\sin x}}$$

$$y(0) = \frac{1}{1-A} = 2 \Rightarrow A = 1/2$$

$$\boxed{y(x) = \frac{1}{1 - \frac{1}{2}e^{\sin x}}}$$

2. (10 points) Find a particular solution of the differential equation

$$y'' + y = \csc x, \quad 0 < x < \pi/2.$$

$y_c = c_1 \cos x + c_2 \sin x \Rightarrow y_1 = \cos x,\ y_2 = \sin x$

Variation of parameter equations!:

$$u_1' \cos x + u_2' \sin x = 0$$
$$-u_1' \sin x + u_2' \cos x = \csc x$$

$\Rightarrow$ ① $u_1' \cos x \sin x + u_2' \sin^2 x = 0$

② $-u_1' \cos x \sin x + u_2' \cos^2 x = \cot(x)$

add ① and ② to get $u_2'(\sin^2 x + \cos^2 x) = \cot(x)$

$\Rightarrow u_2' = \cot(x) \Rightarrow u_2 = \int \frac{\cos(x)}{\sin(x)} = \ln|\sin(x)|$

$u_1' = -u_2' \frac{\sin x}{\cos x} = -1 \Rightarrow u_1' = -1 \Rightarrow u_1 = -x$

$y_p(x) = u_1 y_1 + u_2 y_2 = -x\cos x + \ln|\sin x| \cdot \sin x$

$= \boxed{-x\cos x + \ln(\sin x)\cdot \sin x,\quad 0 < x < \pi/2}$

3. (10 points) Find the interval of convergence, including determination of the convergence at the end points, for the power series

$$\sum_{n=2}^{\infty} \frac{(-2)^n(x+1)^n}{n(\ln n)^2}. \quad = \quad \sum a_n$$

We have

- $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-2)^{n+1}(x+1)^{n+1}}{(n+1)(\ln(n+1))^2} \cdot \frac{n(\ln n)^2}{(-2)^n(x+1)^n} \right|$

$$= \lim_{n\to\infty} \left(|x+1|\, 2 \cdot \frac{n}{n+1} \cdot \left(\frac{\ln n}{\ln(n+1)} \right)^2 \right)$$

- $\lim_{n\to\infty} \frac{n}{n+1} = 1$ and $\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)}$

$$\underset{\text{l'Hospital}}{=} \lim_{n\to\infty} \left(\frac{1/n}{(1/n+1)} \right) = \lim_{n\to\infty} \frac{n+1}{n} = 1,$$

Since $f(x) = x^2$ is continuous at $x = 1$, we have

$$\lim_{n\to\infty} \left(\frac{\ln(n)}{\ln(n+1)} \right)^2 = \left(\lim_{n\to\infty} \left(\frac{\ln(n)}{\ln(n+1)} \right) \right)^2 = 1$$

- So: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 2|x+1|$, which is less than 1 for $|x+1| < 1/2$ ie, $-3/2 < x < -\frac{1}{2}$.

- At $x = -\frac{1}{2}$; $\sum_{n=2}^{\infty} \frac{(-2)^n (\frac{1}{2})^n}{n(\ln n)^2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$, and

$\lim_{n\to\infty} \frac{1}{n(\ln n)^2} = 0$, and $\frac{1}{(n+1)\ln(n+1)^2} < \frac{1}{n(\ln n)^2}$, so converges by AST.

- At $x = -\frac{3}{2}$; $\sum_{n=2}^{\infty} \frac{(-2)^n(-\frac{1}{2})^n}{n(\ln n)^2} = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. We have

$f(x) = \frac{1}{x(\ln x)^2}$ is decreasing, continuous and positive on $[2, \infty)$,

So, since $\int_2^{\infty} f(x)\,dx = \frac{1}{\ln 2}$ is convergent, the series converges by integral test. So, IoC is $[-3/2, -1/2]$

4. (10 points) Consider the function $f(x) = \int_0^x \cos(t^2)\,dt$.

(a) Find the Maclaurin series of $f(x)$.

We have $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

so $\cos x^2 = \sum_{n=0}^{\infty} (-1)^n x^{4n}$

$$f(x) = \int_0^x \cos(t^2)\,dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{(2n)!}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{t^{4n+1}}{4n+1}$$

(b) Estimate the accuracy of the approximation $f(x) \approx T_5(x)$ (the Maclaurin polynomial of degree 5) for $|x| \leq 1/2$.

From part (a), $T_5(x) = x - \frac{x^5}{10}$

Using alternating series test, we have

$$|R_5(x)| \leq \left| \frac{x^9}{4! \cdot 9} \right| = \frac{|x|^9}{4! \cdot 9} \leq \frac{1}{4! \cdot 9 \cdot 2^9}$$

5. (15 points) Evaluate the integral or show that it is divergent (continued on next page).

(a) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$ With $u = \ln x$ $dv = \frac{dx}{x^2}$

$du = \frac{dx}{x}$ $v = -\frac{1}{x}$,

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} -\left[\frac{\ln x}{x}\right]_1^t + \int_1^t \frac{dx}{x^2}$$

$$= \lim_{t\to\infty} -\left(\frac{\ln t}{t} - 0\right) - \left[\frac{1}{x}\right]_1^t = \lim_{t\to\infty} \frac{-\ln t}{t} - \left(\frac{1}{t} - 1\right)$$

$$= \lim_{t\to\infty} \frac{-\ln t}{t} + 1 = \lim_{t\to\infty} -\frac{1}{t} + 1 = 1.$$

(by L'Hospital)

(b) $\displaystyle\int_0^1 \frac{e^{1/x}}{x^3}\,dx$ With $u = \frac{1}{x}$, $du = -\frac{1}{x^2}$, and $e^{\frac{1}{x}} = e^u$,

$$\int_0^1 \frac{e^{\frac{1}{x}}}{x^3}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{e^{\frac{1}{x}}}{x^3} = \lim_{t\to\infty} -\int_t^1 ue^u\,du = \left\{\begin{array}{ll} u' = u & dv = e^u du \\ du' = du & v = e^u \end{array}\right\}$$

$$= \lim_{t\to\infty} -\left[ue^u\right]_t^1 + \int_t^1 e^u\,du = \lim_{t\to\infty} -\left[ue^u\right]_t^1 + \left[e^u\right]_t^1$$

$$= \lim_{t\to\infty} -(e - te^t) + (e - e^t) = \lim_{t\to\infty} te^t - e^t = \lim_{t\to\infty} e^t(t-1).$$

Diverges.

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(c) $\displaystyle\int_0^{\pi/2} \frac{\cos t}{(1+\sin^2 t)^{5/2}}\,dt$ With $u = \sin t$, $du = \cos t$,

$$\int_0^{\frac{\pi}{2}} \frac{\cos t}{\left(1+\sin^2 t\right)^{\frac{5}{2}}}\,dt = \int_0^1 \frac{du}{\left(1+u^2\right)^{\frac{5}{2}}} = \left\{ \begin{array}{l} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{array} \right\}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\left(1+\tan^2\theta\right)^{\frac{5}{2}}} = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^5\theta} = \int_0^{\frac{\pi}{4}} \cos^3\theta\, d\theta = \int_0^{\frac{\pi}{4}} \cos\theta\left(1-\sin^2\theta\right) d\theta$$

$$= \left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}\cdot 3} = \frac{5\sqrt{2}}{12}.$$

6. (15 points) Determine if the series below are absolutely convergent (AC), conditionally convergent (CC), or divergent (D) (continued on next page).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+3n}}$

$$\frac{n}{\sqrt{n^3+3n}} = \frac{1}{\sqrt{n+3/n}} \to 0 \quad \text{as } n \to \infty$$

& is eventually decreasing (can check this by differentiation) so by the AST the series converges.

$$\frac{n/\sqrt{n^3+3n}}{1/\sqrt{n}} = \frac{1}{\sqrt{1+3/n^2}} \to 1 \quad \& \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges, so the series is $\boxed{CC}$ by limit comparison

(b) $\displaystyle\sum_{n=1}^{\infty} n \sin(n^{-3/2})$

$$\frac{n \sin(n^{-3/2})}{n^{-1/2}} = \frac{\sin(n^{-3/2})}{n^{-3/2}} \to 1 \text{ as } n \to \infty$$

(because $\frac{\sin(x)}{x} \to 1$ as $x \to 0$),

& $\displaystyle\sum_{n=1}^{\infty} n^{-1/2}$ diverges, so the series $\boxed{\text{diverges}}$ by limit comparison

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(c) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n}}\right)$

$\sum_{n=1}^{\infty}\left|(-1)^n\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n}}\right)\right| = \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$, which has n-th partial sum $s_n = 1 - \frac{1}{\sqrt{n-1}}$ (it telescopes) Because s_n converges, $\sum_{n=1}^{\infty}\left|(-1)^n\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n}}\right)\right|$ converges, ie. the series is $\boxed{AC}$

7. (15 points) Consider an undamped mass-spring system with mass m and spring constant k, subject to an external force $F(t) = F_0 \cos \omega t$ where $\omega = \sqrt{k/m}$. Find the position $x(t)$ of the mass at time t, relative to the equilibrium position, given that it starts from the position $x(0) = x_0$ with velocity $x'(0) = v_0$.

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos wt$$

$$mr^2 + k = 0 \Rightarrow r = \pm i\sqrt{\frac{k}{m}} = \pm iw$$

$$X_c(t) = C_1 \cos wt + C_2 \sin wt$$

$$X_p(t) = At\cos wt + Bt\sin wt$$

$$X_p'(t) = A\cos wt + B\sin wt - Aw\, t\sin wt + Bwt\cos wt$$

$$X_p''(t) = -Aw\sin wt + Bw\cos wt - Aw\sin wt + Bw\cos wt$$
$$- Aw^2 t\cos wt - Bw^2 t\sin wt$$

$$① \; m(2Bw) = F_0 \qquad ② \; -2Awm = 0$$
$$B = \frac{F_0}{2mw}, \qquad \Rightarrow A = 0.$$

$$\therefore X_p(t) = \frac{F_0}{2mw} \cdot t\sin wt$$

$$\therefore X(t) = X_c(t) + X_p(t) = C_1\cos wt + C_2 \sin wt + \frac{F_0}{2mw} t\sin wt$$

$$X(0) = C_1 = X_0$$

$$X'(0) = C_2 w = V_0 \qquad C_2 = \frac{V_0}{W}$$

$$\therefore X(t) = X_0 \cos wt + \frac{V_0}{W}\sin wt + \frac{F_0}{2mw} t\sin wt.$$

8. (15 points)

(a) Use power series methods to solve the initial value problem
$y'' - xy' - 2y = -4x^2$, $y(0) = 1$, $y'(0) = 1$.

$$\begin{cases} y = \sum_{n=0}^{\infty} c_n x^n, \quad c_0 = 1 \\ xy' = \sum_{n=0}^{\infty} n c_n x^n, \quad c_1 = 1 \\ y'' = \sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n \end{cases}$$

$n=2$: $3 \cdot 4 \cdot c_4 - 2c_2 - 2c_2 = -4$

$$\Rightarrow c_4 = \frac{\cancel{3}\,4c_2 - 4}{3 \cdot 4} = \frac{c_2 - 1}{3}$$

$n \neq 2$: $(n+1)(n+2)c_{n+2} - nc_n - 2c_n = 0$

$$c_{n+2} = \frac{1}{n+1} c_n$$

$c_0 = 1$

$c_2 = \frac{1}{1} c_0 = 1$

$c_4 = \frac{c_2 - 1}{3} = 0$

$c_6 = c_8 = \cdots = 0$

$c_1 = 1$

$c_3 = \frac{1}{2} \cdot c_1 = \frac{1}{2}$

$c_5 = \frac{1}{2 \cdot 4}$

$c_7 = \frac{1}{2 \cdot 4 \cdot 6}$

$\vdots$

$c_{2n+1} = \frac{1}{2 \cdot 4 \cdots (2n)} = \frac{1}{2^n \cdot n!}$

$$\Rightarrow y = 1 + x^2 + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n \cdot n!}$$

(b) Write the solution in terms of elementary functions.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow y = 1 + x^2 + x e^{x^2/2}$$