UCB Math 1B, Fall 2009: Midterm 2, Solutions

Prof. Persson, November 9, 2009

Name: ______________________________

SID: ______________________________

Section: Circle your discussion section below:

Sec	Time	Room	GSI
01	MW 8am - 9am	75 Evans	G. Melvin
02	MW 8am - 9am	5 Evans	T. Wilson
03	MW 10am - 11am	75 Evans	D. Cristofaro-Gardiner
04	MW 10am - 11am	3113 Etcheverry	E. Kim
05	MW 11am - 12pm	81 Evans	G. Melvin
06	MW 12pm - 1pm	5 Evans	T. Wilson
07	MW 1pm - 2pm	2 Evans	A. Tilley
09	MW 2pm - 3pm	247 Dwinelle	D. Cristofaro-Gardiner
10	MW 3pm - 4pm	4 Evans	E. Kim
11	MW 4pm - 5pm	3113 Etcheverry	A. Tilley
12	TT 11:30am - 2pm	230C Stephens	L. Martirosyan

Other/none, explain: ______________________________

Grading

1	/ 5
2	/ 5
3a	/ 5
3b	/ 5
4a	/ 5
4b	/ 5
5	/ 5
	/35

Instructions:

- One double-sided sheet of notes, no books, no calculators.
- Exam time 50 minutes, do all of the problems.
- You must justify your answers for full credit.
- Write your answers in the space below each problem.
- If you need more space, use reverse side or scratch pages. Indicate clearly where to find your answers.

1. (5 points) Find the interval of convergence, including determination of the convergence at the end points, for the power series below.

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n \cdot 3^{2n}}$$

Solution: Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)\cdot 9^{n+1}} \cdot \frac{n \cdot 9^n}{x^n}\right| = \frac{|x|}{9} \cdot \frac{n}{n+1} \to \frac{|x|}{9} \text{ as } n \to \infty$$

$$\frac{|x|}{9} < 1 \iff |x| < 9$$

$$x = 9 : a_n = \frac{(-1)^n}{n} \implies \text{Convergent by the alternating series test}$$

$$x = -9 : a_n = \frac{1}{n} \implies \text{Divergent (p-series with } p = 1)$$

$$\implies I = (-9, 9]$$

2. (5 points) Show that the series

$$y = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)}$$

is a solution of the differential equation

$$y' = 1 + xy.$$

Solution:

$$\begin{aligned}
xy &= \sum_{n=0}^{\infty} \frac{x^{2n+2}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)} \\
y' &= \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)} = \sum_{n=0}^{\infty} \frac{x^{2n}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} \\
&= 1 + \sum_{n=0}^{\infty} \frac{x^{2n+2}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)} \\
&\Longrightarrow y' = 1 + xy
\end{aligned}$$

3. Determine if the series below are absolutely convergent (AC), conditionally convergent (CC), or divergent (D).

a) (5 points) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2-3\sin n}{6}\right)^n$

Solution: Check for absolute convergence by bounding the terms:

$$|a_n| = \left|\frac{2-3\sin n}{6}\right|^n \leq \left(\frac{5}{6}\right)^n$$

Compare with the convergent geometric series $\sum_{n=0}^{\infty}\left(\frac{5}{6}\right)^n$
$\Longrightarrow$ Series is *absolutely convergent* (AC).

b) (5 points) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left[\sin(1/n^2)\right]^{1/3}$

Solution: Convergent by the alternating series test, since $\sin(1/n^2)$ is decreasing for $n \geq 1$ and $\sin(1/n^2) \to \sin 0 = 0$ as $n \to \infty$.

Since $\left[\sin(1/n^2)\right]^{1/3} \approx (1/n^2)^{1/3} = 1/n^{2/3}$, use the limit comparison test with $b_n = 1/n^{2/3}$ to check for absolute convergence:

$$\frac{|a_n|}{b_n} = \frac{\left[\sin(1/n^2)\right]^{1/3}}{1/n^{2/3}} = \left(\frac{\frac{1}{n^2} - \frac{1}{n^6 \cdot 3!} + \frac{1}{n^{10} \cdot 5!} - \cdots}{\frac{1}{n^2}}\right)^{1/3} \to 1 \text{ as } n \to \infty$$

Therefore, the absolute series is divergent, since $\sum_{n=1}^{\infty} 1/n^{2/3}$ is divergent (p-series with $p = 2/3$)
$\Longrightarrow$ Series is *conditionally convergent* (CC).

4. Find the sum of the series below.

a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

Solution: Write the rational function as a partial fraction:

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$$

which gives the partial sum

$$\begin{aligned} s_n &= \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots \\ &+ \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2} \\ &= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \to \frac{3}{2} \text{ as } n \to \infty \end{aligned}$$

b) (5 points) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{1+3\cdot(-1)^n}\right)^n$

Solution: Split the series:

$$\begin{aligned} n \text{ even } &: a_n = \frac{1}{4^n} \\ n \text{ odd } &: a_n = \frac{1}{(-2)^n} = -\frac{1}{2^n} \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} a_n &= \sum_{n=0}^{\infty} \left[\frac{1}{4^{2n}} - \frac{1}{2^{2n+1}}\right] = \sum_{n=0}^{\infty} \left[\frac{1}{16^n} - \frac{1}{2}\cdot\frac{1}{4^n}\right] \\ &= \frac{1}{1-1/16} - \frac{1}{2}\cdot\frac{1}{1-1/4} = \frac{16}{15} - \frac{2}{3} = \frac{2}{5} \end{aligned}$$

5. (5 points) Find all x that satisfy the equation

$$\sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n+2} = \frac{2}{9}.$$

Solution:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n \qquad |x| < 1$$

$$\frac{x^2}{(1+x^2)^2} = \sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n+2} \qquad |x^2| < 1$$

$$\frac{x^2}{(1+x^2)^2} = \frac{2}{9} \Longrightarrow 9x^2 = 2(1+2x^2+x^4) \Longrightarrow x^4 - \frac{5}{2}x^2 + 1 = 0$$

$$\Longrightarrow x^2 = \frac{5}{4} \pm \sqrt{\frac{25}{16} - 1} = \frac{5 \pm 3}{4} = \frac{1}{2}, 2$$

But the series is only convergent for $|x^2| < 1$, and therefore

$$x = \pm\frac{1}{\sqrt{2}}$$