UCB Math 1B, Fall 2009: Midterm 1, Solutions

Prof. Persson, October 5, 2009

Name: ______________________________

SID: ______________________________

Section: Circle your discussion section below:

Sec	Time	Room	GSI
01	MW 8am - 9am	75 Evans	G. Melvin
02	MW 8am - 9am	5 Evans	T. Wilson
03	MW 10am - 11am	75 Evans	D. Cristofaro-Gardiner
04	MW 10am - 11am	3113 Etcheverry	E. Kim
05	MW 11am - 12pm	81 Evans	G. Melvin
06	MW 12pm - 1pm	5 Evans	T. Wilson
07	MW 1pm - 2pm	2 Evans	A. Tilley
09	MW 2pm - 3pm	247 Dwinelle	D. Cristofaro-Gardiner
10	MW 3pm - 4pm	4 Evans	E. Kim
11	MW 4pm - 5pm	3113 Etcheverry	A. Tilley
12	TT 11:30am - 2pm	230C Stephens	L. Martirosyan

Other/none, explain: ______________________________

Grading

1	/ 5
2	/ 5
3a	/ 3
3b	/ 3
3c	/ 4
4a	/ 5
4b	/ 5
5	/ 5
	/35

Instructions:

- One double-sided sheet of notes, no books, no calculators.
- Exam time 50 minutes, do all of the problems.
- You must justify your answers for full credit.
- Write your answers in the space below each problem.
- If you need more space, use reverse side or scratch pages. Indicate clearly where to find your answers.

1. (5 points) Evaluate the integral:

$$\int \arctan x \, dx$$

Solution: Integrate by parts:

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2} \, dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C$$

2. (5 points) Determine if the integral below is convergent or divergent. If it is convergent, evaluate it.

$$\int_e^\infty \frac{1}{x(\ln x)^3} \, dx$$

Solution:

$$\int_e^\infty \frac{1}{x(\ln x)^3} \, dx = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^3} \, dx = \begin{bmatrix} u = \ln x, & x = e \leftrightarrow u = 1 \\ du = \frac{1}{x} \, dx, & x = t \leftrightarrow u = \ln t \end{bmatrix}$$

$$= \lim_{t\to\infty} \int_1^{\ln t} \frac{1}{u^3} \, du = \lim_{t\to\infty} \left[\frac{u^{-2}}{-2} \right]_1^{\ln t}$$

$$= -\frac{1}{2} \lim_{t\to\infty} \left(\frac{1}{(\ln t)^2} - 1 \right) = \frac{1}{2} \quad \Longrightarrow \quad \text{Convergent}$$

3. Determine if the sequences $\{a_n\}$ below converge or diverge.
If they converge, find the limit.

a) (3 points) $a_n = e^{1/n}$

Solution:

$$\lim_{n\to\infty} e^{1/n} = e^{\lim_{n\to\infty}(1/n)} = e^0 = 1 \quad \text{(since } e^x \text{ continuous at } x = 0\text{)}$$

$$\implies \quad \text{Convergent}$$

b) (3 points) $a_n = \dfrac{(2n-1)!}{(2n+1)!}$

Solution:

$$a_n = \frac{(2n-1)!}{(2n+1)!} = \frac{1\cdot 2\cdots(2n-1)}{1\cdot 2\cdots(2n-1)(2n)(2n+1)}$$
$$= \frac{1}{(2n)(2n+1)} \to 0 \text{ as } n\to\infty \quad \implies \quad \text{Convergent}$$

c) (4 points) $a_n = \dfrac{(-1)^n n^3}{n^3 + 2n^2 + 1}$

Solution:

$$|a_n| = \frac{1}{1 + 2/n + 1/n^3} \to 1 \text{ as } n \to \infty$$

But $\{a_n\}$ alternates in sign $\implies$ its terms do not approach a single real number $\implies$ Divergent

4. Consider the region bounded by the curves $y = 0$, $x = -1$, $x = 1$, and

$$y = \frac{1}{(x^2+1)^{3/2}}.$$

a) (5 points) Find the area A of the region. Hint: Use trigonometric substitution.

Solution:

$$A = \int_{-1}^{1} \frac{1}{(x^2+1)^{3/2}}\,dx = \begin{bmatrix} x = \tan\theta \\ dx = \sec^2\theta\,d\theta \\ x = \pm 1 \leftrightarrow \theta = \pm\frac{\pi}{4} \end{bmatrix} = \int_{-\pi/4}^{\pi/4} \frac{1}{\sec^3\theta}\sec^2\theta\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \cos\theta\,d\theta = [\sin\theta]_{-\pi/4}^{\pi/4} = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

b) (5 points) Find the centroid of the region.

Solution:

$\bar{x} = 0$ by symmetry.

$$\bar{y} = \frac{1}{2A}\int_{-1}^{1} \frac{1}{(x^2+1)^3}\,dx = \frac{1}{2A}\int_{-\pi/4}^{\pi/4} \frac{1}{\sec^6\theta}\sec^2\theta\,d\theta = \frac{1}{2A}\int_{-\pi/4}^{\pi/4} \cos^4\theta\,d\theta$$

$$= \frac{1}{2A}\int_{-\pi/4}^{\pi/4} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{1}{8A}\int_{-\pi/4}^{\pi/4} \left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{1}{8A}\int_{-\pi/4}^{\pi/4} \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{8A}\left[\frac{3}{2}\theta + \sin 2\theta + \frac{\sin 4\theta}{8}\right]_{-\pi/4}^{\pi/4} = \frac{1}{8A}\left[\frac{3\pi}{4} + 2 + 0\right]$$

$$= \frac{3\pi+8}{32\sqrt{2}} = \frac{\sqrt{2}(3\pi+8)}{64}$$

5. (5 points) If f is a quadratic function $f(x) = ax^2 + bx + 1$, and

$$\int \frac{f(x)}{x^2(x+1)^3}\,dx$$

is a rational function, find the value of b.

Solution:

$$\frac{ax^2+bx+1}{x^2(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

Integral a rational function $\implies$ No logarithms $\implies$ $A = C = 0$

$$\begin{aligned} ax^2 + bx + 1 &= B(x+1)^3 + Dx^2(x+1) + Ex^2 \\ &= B(x^3 + 3x^2 + 3x + 1) + D(x^3 + x^2) + Ex^2 \\ &\implies \begin{cases} 1 = B \\ b = 3B \end{cases} \implies b = 3 \end{aligned}$$