ME 104: ENGINEERING MECHANICS II
Department of Mechanical Engineering
University of California at Berkeley

Spring Semester 2009

Professor Oliver M. O'Reilly

Final Examination
Monday, May 18, 2009
5:00 pm to 8:00 pm
234 Hearst Gym

Closed Books and Closed Notes
Each Question is Worth 25 Points

Useful Formulae

For all the corotational bases shown in the figures

$$\begin{aligned} \mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x. \end{aligned} \tag{1}$$

The following identity for the angular momentum of a rigid body relative to a point P will also be useful:

$$\mathbf{H}_P = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_P) \times m\bar{\mathbf{v}}. \tag{2}$$

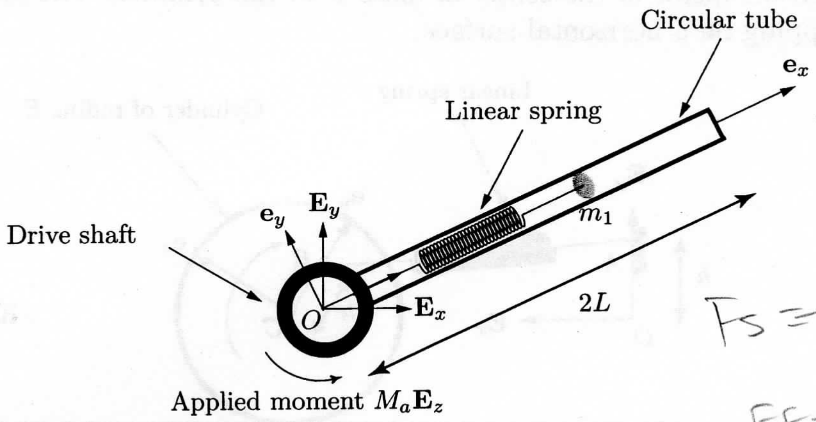
Finally, recall that the work-energy theorem of a rigid body which is subject to a system of K forces and a pure moment $\mathbf{M}_p$ is

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}. \tag{3}$$

Here, $\mathbf{v}_i$ is the velocity vector of the point where the force $\mathbf{F}_i$ is applied.

Question 1
A Particle in a Tube

As shown in Figure 1, a particle of mass m_1 is free to move inside a long tube. The particle is connected to a fixed point O by a linear spring of stiffness K and unstretched length L_0. The end of the tube at O is fixed to a drive shaft which is spun with an angular speed $\Omega(t)$. The contact between the particle and the tube is smooth. In addition to normal forces, a vertical gravitational force $-m_1 g\mathbf{E}_z$ acts on the particle. The tube has a mass m_2, moment of inertia (relative to its center of mass) I_{zz}, and a length $2L$.



$F_s = k(\|\vec{r} - \vec{r}_A - L_0\|)$

$F_s = k(\|\vec{r} - \vec{r}_A\| - L_0)\frac{\vec{r} - \vec{r}_A}{\|\vec{r} - \vec{r}_A\|}$

$F_f = -\mu_d \|N\| \frac{\vec{v}_P}{\|\vec{v}_P\|}$

Figure 1: A particle of mass m_1 is attached to a fixed point by a linear spring. The particle moves inside a long tube which is being rotated about O with an angular velocity $\Omega\mathbf{E}_z$.

(a) *(3 Points)* Assuming that the position vector $\mathbf{x}_1 = x\mathbf{e}_x$ of the particle and the position vector of the center of mass of the rod is $\bar{\mathbf{x}}_2 = L\mathbf{e}_x$, show that the linear momentum of the system is

$$\mathbf{G} = m_1\dot{x}\mathbf{e}_x + (m_1 x + m_2 L)\,\dot{\theta}\mathbf{e}_y, \tag{4}$$

where $\dot{\theta} = \Omega$.

(b) *(6 Points)* Establish expressions for the kinetic energy of the system and the angular momentum of the system relative to the point O.

(c) *(6 Points)* Draw free-body diagrams of the system, the tube, and the particle. Give clear expressions for the spring force and normal force.

(d) *(5 Points)* Show that the equation governing the motion of the particle is

$$m_1\ddot{x} - m_1 x\Omega^2 = -K(x - L_0). \tag{5}$$

In addition, show that the moment $M_a\mathbf{E}_z$ which is applied to rotate the tube is

$$M_a = \left(I_{zz} + m_2L^2 + m_1x^2\right)\dot{\Omega} + 2m_1x\dot{x}\Omega. \tag{6}$$

(e) *(5 Points)* Starting from the work-energy theorem show that the power $\mathcal{P}$ needed to rotate the tube is

$$\mathcal{P} = \frac{d}{dt}\left(\frac{1}{2}\left(I_{zz} + m_2L^2 + m_1x^2\right)\Omega^2 + \frac{1}{2}m_1\dot{x}^2 + \frac{K}{2}(x - L_0)^2\right). \tag{7}$$

$\frac{1}{2}I\omega^2$ $\frac{1}{2}mv^2$ $\frac{1}{2}Kx^2$

Question 2

A Rolling Rigid Body

As shown in Figure 2, a circular cylinder of radius R, mass m, and moment of inertia (relative to its center of mass) I_{zz} is attached to a fixed point A by a linear spring of stiffness K and unstretched length L_0. The other end of the spring is attached to a point B on the cylinder. The position vectors of A and B have the representations

$$\mathbf{x}_A = h\mathbf{E}_y, \qquad \mathbf{x}_B = \bar{\mathbf{x}} + h\mathbf{e}_y, \tag{8}$$

where $\bar{\mathbf{x}}$ is the position vector of the center of mass C of the cylinder. The cylinder is assumed to roll without slipping on a horizontal surface.

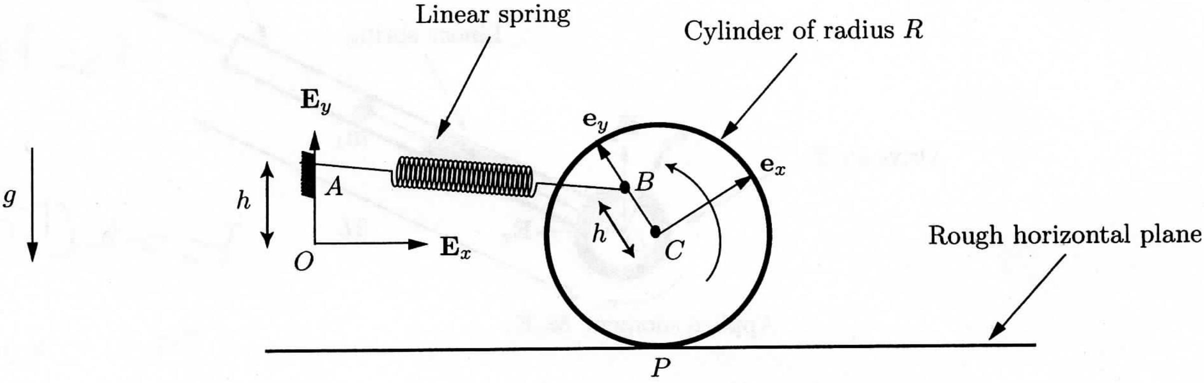


Figure 2: A rigid cylinder of mass m and radius R is attached to a fixed point A by a linear spring. The body rolls without slipping on a horizontal surface.

(a) *(5 Points)* Starting from the result $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$ for any two points on a rigid body, show that the velocity vectors of the center of mass C and the instantaneous point of contact P have the representations

$$\bar{\mathbf{v}} = \dot{x}\mathbf{E}_x, \qquad \mathbf{v}_P = \left(\dot{x} + R\dot{\theta}\right)\mathbf{E}_x, \tag{9}$$

where $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ is the angular velocity of the rigid body.

(b) *(5 Points)* Draw a free-body diagram of the rolling rigid body. Give a clear expression for the spring force.

ANSWER EITHER (c) OR (d):

(c) *(15 Points)* Using balances of linear and angular momentum, show that the equation governing the motion of the cylinder is

$$\left(I_{zz} + mR^2\right)\ddot{\theta} = \frac{K\epsilon}{\epsilon + L_0}\left(x\left(R + h\cos(\theta)\right) - h\sin(\theta)\left(h + R\right)\right), \tag{10}$$

where ϵ is the extension of the spring.

(d) *(15 Points)* Consider the case where $h = 0$. Using balances of linear and angular momentum, show that rolling can occur provided

$$\left|\left(\frac{I_{zz}}{I_{zz} + mR^2}\right)\frac{K\epsilon x}{\epsilon + L_0}\right| \leq \mu_s mg, \qquad \text{where} \quad \epsilon = |x| - L_0, \tag{11}$$

and μ_s is the coefficient of static friction. Using (11), determine the range of values of x for which slipping will occur.

Question 3

An Imbalanced Shaft

As shown in Figure 3, a shaft of mass m_1, radius R and length $2L$ is imbalanced by the presence of two rigidly attached particles, each of mass $\frac{m_2}{2}$. The shaft is driven into rotation about the $\mathbf{E}_z$ axis by an applied torque $T_a\mathbf{E}_z$.

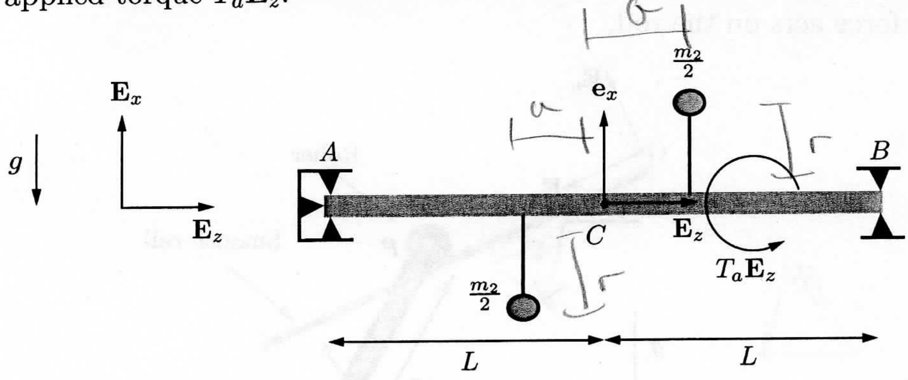


Figure 3: An imbalanced shaft-particle system which is free to rotate about the $\mathbf{E}_z$ axis and is supported by bearings at A and B.

The center of mass of the shaft and the shaft-particle system are coincident and are both assumed to be stationary. The angular momentum of the shaft relative to its center of mass C is

$$I_{zz}\dot{\theta}\mathbf{E}_z, \tag{12}$$

and the position vectors of the particles relative to the center of mass C have the representations

$$\mathbf{x}_1 - \bar{\mathbf{x}} = a\mathbf{E}_z + r\mathbf{e}_x, \qquad \mathbf{x}_2 - \bar{\mathbf{x}} = -a\mathbf{E}_z - r\mathbf{e}_x. \tag{13}$$

(a) *(8 Points)* Show that the angular momentum $\mathbf{H}$ of the system relative to the center of mass C of the system has the representation

$$\mathbf{H} = \left(I_{zz} + m_2r^2\right)\dot{\theta}\mathbf{E}_z - m_2ar\dot{\theta}\mathbf{e}_x. \tag{14}$$

(b) *(5 Points)* Draw a free-body diagram of the system.

(c) *(5 Points)* Show that the angular speed of the shaft is governed by the equation

$$\left(I_{zz} + m_2r^2\right)\ddot{\theta} = T_a. \tag{15}$$

By varying T_a and measuring $\theta(t)$, is it possible to detect the imbalance?

(d) *(7 Points)* Assuming that $\dot{\theta}$ is constant, *Verify* that the bearing forces are

$$\begin{aligned}
\mathbf{R}_A &= \left(\frac{m_1 + m_2}{2}\right)g\mathbf{E}_x - \left(\frac{-m_2ar\omega_0^2}{2L}\right)\mathbf{e}_x, \\
\mathbf{R}_B &= \left(\frac{m_1 + m_2}{2}\right)g\mathbf{E}_x + \left(\frac{-m_2ar\omega_0^2}{2L}\right)\mathbf{e}_x,
\end{aligned} \tag{16}$$

where $\dot{\theta} = \omega_0$.

Question 4

A Rigid Body Sliding on a Smooth Rail

As shown in Figure 4, a rod of mass m, moment of inertia (relative to its center of mass) I_{zz} and length $2L$ is connected by a roller of negligible mass and inertia at a point P. The roller is free to move on a smooth rail. The rail is inclined at an angle ϕ to the horizontal. A vertical gravitational force acts on the rod.

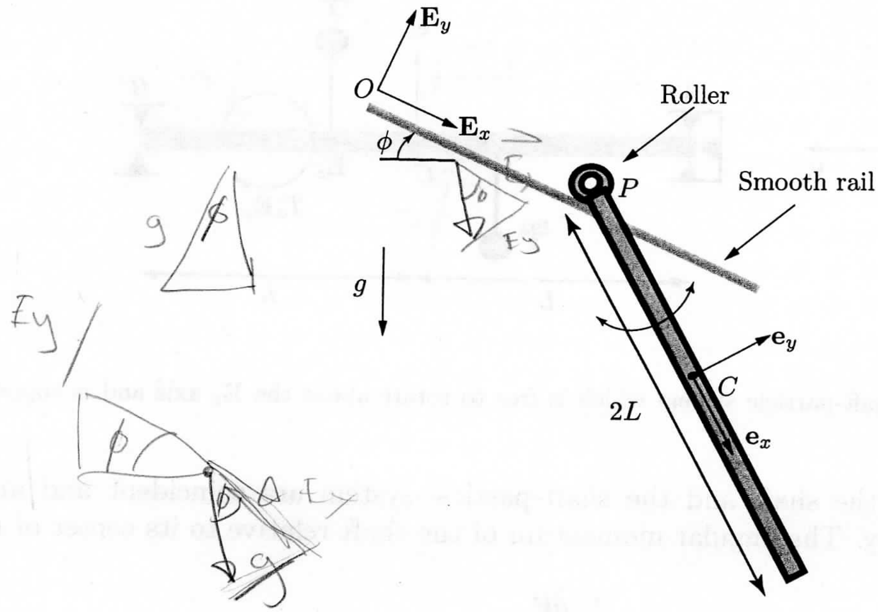


Figure 4: A rod of mass m which is moves on a smooth rail. The point P of the rod is free to translate along the rail and the rod is free to rotate about P.

(a) *(3 Points)* If the position vectors of the center of mass C of the rod and the point P are

$$\bar{\mathbf{x}} = L\mathbf{e}_x + \mathbf{x}_P, \qquad \mathbf{x}_P = x\mathbf{E}_x, \tag{17}$$

respectively, then show that the velocity vector of the center of mass has the representation

$$\bar{\mathbf{v}} = \left(-L\dot{\theta}\sin(\theta) + \dot{x}\right)\mathbf{E}_x + L\dot{\theta}\cos(\theta)\,\mathbf{E}_y. \tag{18}$$

(b) *(7 Points)* Show that the angular momentum of the rod relative to P and the kinetic energy T have the representations

$$\begin{aligned} \mathbf{H}_P &= \left(I_{zz} + mL^2\right)\dot{\theta}\mathbf{E}_z - mL\dot{x}\sin(\theta)\,\mathbf{E}_z, \\ T &= \frac{1}{2}\left(\left(I_{zz} + mL^2\right)\dot{\theta}^2 + m\dot{x}^2 - 2m\dot{x}L\dot{\theta}\sin(\theta)\right). \end{aligned} \tag{19}$$

(c) *(3 Points)* Draw a free-body diagram of the rod.

(d) *(7 Points)* Show that the equations governing the motion of the rigid body are

$$\begin{aligned} m\ddot{x} - mL\ddot{\theta}\sin(\theta) - mL\dot{\theta}^2\cos(\theta) &= mg\sin(\phi), \\ \left(I_{zz} + mL^2\cos^2(\theta)\right)\ddot{\theta} - mL\dot{\theta}^2\cos(\theta)\sin(\theta) &= -mgL\cos(\theta)\cos(\phi). \end{aligned} \tag{20}$$

(e) *(5 Points)* Give a clear explanation of why $\dot{\mathbf{H}}_P \neq \mathbf{M}_P$, where $\mathbf{M}_P$ is the resultant moment relative to P.

because Ho is non-fixed nor center of mass, angular momentum is not conserved