Department of Mathematics, University of California, Berkeley

Math 1A

Alan Weinstein, Fall 2000

Final Examination, Thursday, December 14, 2000

Instructions. Be sure to write on the front cover of your blue book: (1) your name, (2) your Student ID Number, (3) your TA's name (Eric Antokoletz, Victor Deletang, Matthieu Hamel, Andre Henriques, Di-An Jan, Chu-Wee Lim, Russell O'Connor, Alf Onshuus, Emmanuel Py, Shahed Sharif, Dan Stevens, Karla Westphal, or Alexander Woo).

Read the problems very carefully to be sure that you understand the statements. Show all your work as clearly as possible, and circle each final answer to each problem. **When doing a computation, don't put an "=" sign between things which are not equal. When giving explanations, write complete sentences.**

Note: In solving the problems on this exam, you may not use advanced techniques (such as integration by parts) which you may have learned in a previous calculus course. If you have any question about what you are permitted to use on this exam, please ask one of the proctors.

1. [15 points] A driver travelling the 60 mile stretch of highway from Bluesville to Greenville computes the average speed in miles per hour for each 10 mile segment of the trip and finds these averages to be 20, 30, 40, 60, 40, and 20.
(a) What is the total time required for the trip?
(b) What is the average speed for the entire trip? (This is the usual average speed, the "average with respect to time".)
(c) The state highway department decides to post an electronic sign in Bluesville which will tell drivers the expected time for a trip to Greenville. They place a large number of sensors along the highway so that, at any moment, they can accurately estimate the function $s(x)$ which gives the speed (in miles per hour) of the traffic which is currently at a distance x miles from Bluesville (and going in the direction of Greenville). Under the assumption that traffic conditions are steady, so that the function $s(x)$ does not change during the time it takes to make the trip, a computer uses this data to calculate the expected travel time from Bluesville to Greenville as an integral, and the result is displayed on the sign.
What is this integral? In other words, write an integral which expresses in terms of $s(x)$ the expected travel time T (in hours) for the 60 mile journey. You must justify your answer. [*Hint:* Check the units in your answer. Also check to see that your answer is correct when the function $s(x)$ is constant.]

2. [15 points] Let $f(x) = x^4 + \frac{1}{2}x + g(x)$, where $g(x)$ is a differentiable function such that $g(-1) = g(0) = g(1) = 0$. Using a theorem or theorems about continuous and/or differentiable functions, prove that there is at least one value of x for which $f'(x) = 0$.

3. [15 points] (a) Use a linear approximation to $f(x) = x^{1/4}$ around $x_0 = 16$ to estimate the 4th root of 16.32.
(b) Is this estimate greater than or less than the exact 4th root. Why?
(c) Use a single iteration of Newton's method to find an approximate solution of the equation $x^4 - 16.32 = 0$, starting with the initial guess $x = 2$.

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4. [15 points] Evaluate each of the following definite integrals. [*Hint*: look for shortcuts on some of these problems. Don't just look for antiderivatives.]

(a)

$$\int_0^{\sqrt{\pi}} x \sin(x^2) \, dx$$

(b)

$$\int_{-1}^1 \frac{\sin x}{e^x + e^{-x}} \, dx$$

(c)

$$\int_0^1 \frac{d}{dx} \left(e^{4x} \ln\left(\frac{x+2}{x+1}\right)\right) dx$$

(d)

$$\int_0^1 \frac{dt}{1+t^2}$$

(e)

$$\int_0^3 \sqrt{9 - u^2} \, du$$

5. [15 points] (a) Find the area of the region bounded by the curve $y = x^4$ and the line $y = 16$ in two ways, first by integrating with respect to x, and then by integrating with respect to y. Check that the two answers are equal.

(b) Find the number r for which the vertical line $x = r$ divides the region in part (a) into two pieces of equal area.

6. [15 points] Do the following computations.

(a) Find

$$\sum_{i=1}^{50} (3^i - 3^{i-1}).$$

(b) Find

$$\lim_{s \to 0} \frac{1}{s} \int_3^{3+s} \ln t \, dt.$$

(c) Find

$$\frac{d}{d\theta}(\sin(2^\theta)).$$

(d) Find

$$\lim_{n \to \infty} n^2 \left(\frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} + \cdots \frac{1}{(n+n)^3} \right).$$

[*Hint*: write the expression as $1/n$ times a sum. Be careful with the algebra.]

7. [15 points] Let $a > 1$. Two horizontal lines are drawn in the (x, y) plane, one through the point $P = (0, a)$ and one through the two points on the parabola $y = x^2$ which are closest to P. What is the distance between the two lines?

8. [15 points] Find the volume of the solid of revolution obtained by revolving about the x-axis the region bounded by the lines $x = 2$, $x = 6$, and $y = 0$ and the curve $y = \frac{1}{\sqrt{x}}$.

(MATH 1A, Alan Weinstein, Fall 2000, final exam)