Mathematics 1A, Section 2 December 13, 2001
Sarason

Final Examination

SOLUTIONS

Name (printed): ______________________

Signature: ______________________

SID: ______________________

GSI: ______________________

Section Time: ______________________

Put your name on every page.

Closed book except for crib sheet. No calculators.

SHOW YOUR WORK. Cross out anything you have written that you do not want the grader to consider.

The points for each problem are in parentheses. Perfect score = 200.

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Total	
Grade Points	

0. (10) Suppose $f(x) = 1 + x^2$. If $g = f \circ f$ and $h = f \circ g$, what are $g'(1)$ and $h'(1)$?

We have $f'(x) = 2x$. By the chain rule,

$$g'(1) = f'(f(1))f'(1) = f'(2)f'(1) = (4)(2) = 8$$

Also $g(1) = f(f(1)) = f(2) = 5$. By the chain rule, again,

$$h'(1) = f'(g(1))g'(1) = f'(5)g'(1) = (10)(8) = 80$$

1. (15) Find f' for the given functions f.

(a) $f(x)=\sqrt{1+\sqrt{1+x^2}}$ (b) $f(x)=e^{\sqrt{\ln x}}$ (c) $f(x)=\int_0^x e^{-(x-t)^2}\,dt$

(a) By the chain rule,

$$f'(x) = \frac{1}{2\sqrt{1+\sqrt{1+x^2}}}\,\frac{d}{dx}\left(\sqrt{1+x^2}\right) = \frac{x}{2\sqrt{1+x^2}\,\sqrt{1+\sqrt{1+x^2}}}$$

(b) By the chain rule,

$$F'(x) = e^{\sqrt{\ln x}}\,\frac{d}{dx}\left(\sqrt{\ln x}\right) = \frac{e^{\sqrt{\ln x}}}{2\sqrt{\ln x}}\,\frac{d}{dx}(\ln x) = \frac{e^{\sqrt{\ln x}}}{2x\sqrt{\ln x}}$$

(c) In the integral, make the substitution $t = x - u$ to get

$$f(x) = -\int_x^0 e^{-u^2}\,du = \int_0^x e^{-u^2}\,du.$$

By the fundamental theorem of calculus, $f'(x) = e^{-x^2}$.

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2. (15) Evaluate the limits.

(a) $\lim_{x\to 0} \dfrac{\cos 5x - 1}{x^2}$ (b) $\lim_{x\to\infty} x(\sqrt{x^2+1} - x)$ (c) $\lim_{x\to 0} \dfrac{x^x - 1}{\sqrt{x}}$

(a) L'Hospital's rule applies since the numerator and denominator both have the limit 0 at 0:

$$\lim_{x\to 0} \frac{\cos 5x - 1}{x^2} = \lim_{x\to 0} \frac{-5\sin 5x}{2x} = \lim_{x\to 0} \frac{-25\cos 5x}{2} \quad \text{(by l'Hospital, again)}$$

$$= \frac{-25}{2}$$

(b) $$\lim_{x\to\infty} x\left(\sqrt{x^2+1} - x\right) = \lim_{x\to\infty} \frac{x\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\sqrt{x^2+1} + x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1} + x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}} + 1} = \frac{1}{2}$$

(c) Note that $x^x = e^{x\ln x} \longrightarrow 1$ as $x \to 0$ since

$$\lim_{x\to 0} (x\ln x) = \lim_{x\to 0} \frac{\ln x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = 0 \quad \text{(by l'Hospital).}$$

We can thus use l'Hospital's rule on the original limit:

$$\lim_{x\to 0} \frac{x^x - 1}{\sqrt{x}} = \lim_{x\to 0} \frac{e^{x\ln x} - 1}{\sqrt{x}} = \lim_{x\to 0} \frac{e^{x\ln x}(\ln x + 1)}{1/2\sqrt{x}}$$

$$= \lim_{x\to 0} \frac{\ln x + 1}{1/2\sqrt{x}} \quad \left(\text{since } e^{x\ln x} \longrightarrow 1 \text{ as } x \longrightarrow 0\right)$$

$$= \lim_{x\to 0} \frac{1/x}{-1/4x^{3/2}} \quad \text{(l'Hospital again)}$$

$$= \lim_{x\to 0} \frac{-\sqrt{x}}{4} = 0$$

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3. (15) What are the maximum and minimum values of the function $f(x) = (2x+1)e^{-x^2}$ on the interval $[0, 1]$?

We have $f'(x) = 2e^{-x^2} - 2x(2x+1)e^{-x^2} = -2(2x^2+x-1)e^{-x^2}$

$= -2(2x-1)(x+1)e^{-x^2}$

The critical points are $\frac{1}{2}$ and -1. The function f is increasing on $(-1, \frac{1}{2})$ and decreasing on $[\frac{1}{2}, \infty]$. Its maximum on $[0,1]$ therefore occurs at $x = \frac{1}{2}$ and its value is $2e^{-1/4}$.

The minimum of f on $[0,1]$ occurs either at 0 or at 1, so it is either 1 or $3e^{-1}$. Since ~~[illegible]~~ $e < 3$, we have $3e^{-1} > 1$, so the minimum value is 1.

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4. (15) Perform the integrations.

(a) $\int e^x \cos(e^x)dx$ (b) $\int \frac{x^2+x+1}{x-1}dx$ (c) $\int_0^1 (1-x^{1/3})^{3/2}dx$

(a) Make the substitution $u = e^x$, $du = e^x dx$, to get

$$\int e^x \cos(e^x)\, dx = \int \cos u \, du = \sin u + C = \sin(e^x) + C$$

(b) Make the substitution $u = x-1$ to get

$$\int \frac{x^2+x+1}{x-1} dx = \int \frac{(u+1)^2+(u+1)+1}{u} du = \int \frac{u^2+3u+3}{u} du$$

$$= \int \left(u+3+\frac{3}{u}\right) du = \frac{u^2}{2} + 3u + 3\ln|u| + C$$

$$= \frac{(x-1)^2}{2} + 3(x-1) + 3\ln|x-1| + C$$

(c) Make the substitution $1-x^{1/3} = u^2$, $x = (1-u^2)^3$, $dx = -6u(1-u^2)^2$, to get

$$\int_0^1 (1-x^{1/3})^{3/2} dx = \int_1^0 u^3(-6u)(1-u^2)^2 du$$

$$= 6\int_0^1 u^4(1-u^2)^2 = 6\int_0^1 (u^4 - 2u^6 + u^8)\, du$$

$$= 6\left(\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right)\Bigg|_0^1 = 6\left(\frac{1}{5} - \frac{2}{7} + \frac{1}{9}\right)$$

$$= \frac{6(63-90+35)}{315} = \frac{48}{315} = \frac{16}{105}$$

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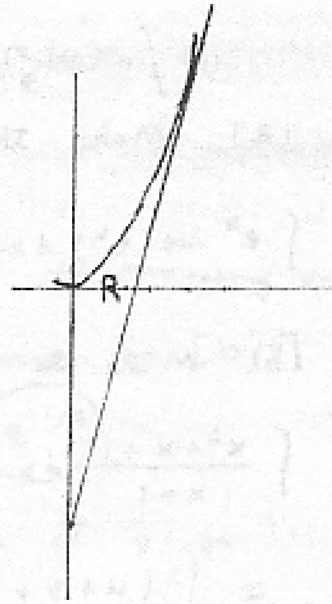
5. (15) Find the area of the region bounded by the parabola $y = x^2$, the y-axis, and the tangent line to the parabola at the point $(2, 4)$.

The tangent line at $(2,4)$ has slope 4, so its equation is $y = 4 + 4(x-2) = 4x - 4$. Hence



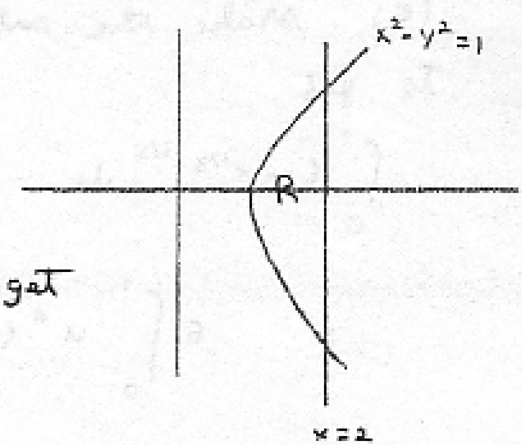
$$\text{area}(R) = \int_0^2 (x^2 - 4x + 4)\,dx$$

$$= \left(\frac{x^3}{3} - 2x^2 + 4x\right)\Big|_0^2 = \frac{8}{3} - 8 + 8$$

$$= \frac{8}{3}$$

6. (15) What is the volume of the solid one obtains by revolving about the y-axis the region bounded by the right branch of the hyperbola $x^2 - y^2 = 1$ and the line $x = 2$?

The region R being revolved lies above and below the interval $[1, 2]$, is bounded above by the curve $y = \sqrt{x^2-1}$, and is bounded below by the curve $y = -\sqrt{x^2-1}$. Using the shell method, for the volume of the generated solid S we get



$$\text{vol}(S) = 2\pi \int_1^2 x\left(\sqrt{x^2-1} - \left(-\sqrt{x^2-1}\right)\right)dx$$

$$= 4\pi \int_1^2 x\sqrt{x^2-1}\,dx \qquad u = x^2-1,\quad du = 2x\,dx$$

$$= 2\pi \int_0^3 \sqrt{u}\,du = 2\pi \left(\frac{u^{3/2}}{3/2}\right)\Big|_0^3 = \frac{4\pi\, 3^{3/2}}{3}$$

$$= 4\pi\sqrt{3}$$

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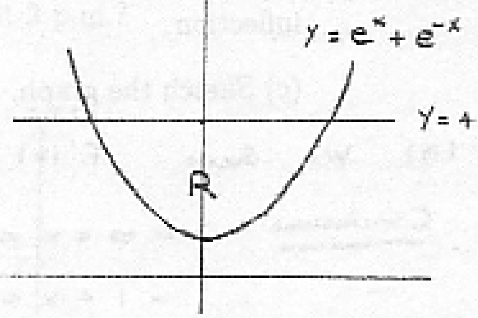
7. (20) What is the volume of the solid one obtains by revolving about the x-axis the region below the line $y = 4$ and above the curve $y = e^x + e^{-x}$?

To obtain the x-coordinates of the points of intersection of $y = 4$ and $y = e^x + e^{-x}$, we solve

$$e^x + e^{-x} = 4$$



for x. The equation can be rewritten as

$$e^{2x} - 4e^x + 1 = 0.$$

The quadratic formula gives $e^x = 2 \pm \sqrt{3}$, so the x-coordinates of the points of intersection are $\pm \ln(2+\sqrt{3})$. (Note that $2+\sqrt{3}$ and $2-\sqrt{3}$ are reciprocals.) The region R to be revolved thus lies over the interval $[-\ln(2+\sqrt{3}), \ln(2+\sqrt{3})]$. For the volume of the generated solid S, the disk method gives

$$\mathrm{vol}(S) = \pi \int_{-\ln(2+\sqrt{3})}^{\ln(2+\sqrt{3})} \left[4^2 - (e^x + e^{-x})^2\right] dx = 2\pi \int_0^{\ln(2+\sqrt{3})} (16 - e^{2x} - 2 - e^{-2x})\, dx$$

$$= 2\pi \int_0^{\ln(2+\sqrt{3})} (14 - e^{2x} - e^{-2x})\, dx = 2\pi \left(14x - \frac{e^{2x}}{2} + \frac{e^{-2x}}{2}\right)\Big|_0^{\ln(2+\sqrt{3})}$$

$$= 2\pi \left(14 \ln(2+\sqrt{3}) - \frac{(2+\sqrt{3})^2}{2} + \frac{(2-\sqrt{3})^2}{2}\right)$$

$$= 28\pi \ln(2+\sqrt{3}) - 8\pi\sqrt{3}$$

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8. (30) For the function $f(x) = \dfrac{x}{(1-x)^2}$:

(a) Determine the intervals of increase, the intervals of decrease, and the local maxima and minima.

(b) Determine the intervals of upward and downward concavity, and the points of inflection.

(c) Sketch the graph.

(a) We have $f'(x) = \dfrac{1}{(1-x)^2} + \dfrac{2x}{(1-x)^3} = \dfrac{1+x}{(1-x)^3}$.

Conclusions

$-\infty < x < -1$: $f' < 0$, & f is decreasing

$-1 < x < 1$: $f' > 0$, & f is increasing

$1 < x < \infty$: $f' < 0$, & f is decreasing

local max. : none

local min. : $x = -1$, $f(-1) = -\frac{1}{4}$

(b) $f''(x) = \dfrac{1}{(1-x)^3} + \dfrac{3(1+x)}{(1-x)^4} = \dfrac{4+2x}{(1-x)^4}$

Conclusions

$-\infty < x < -2$: $f'' < 0$, f is concave down

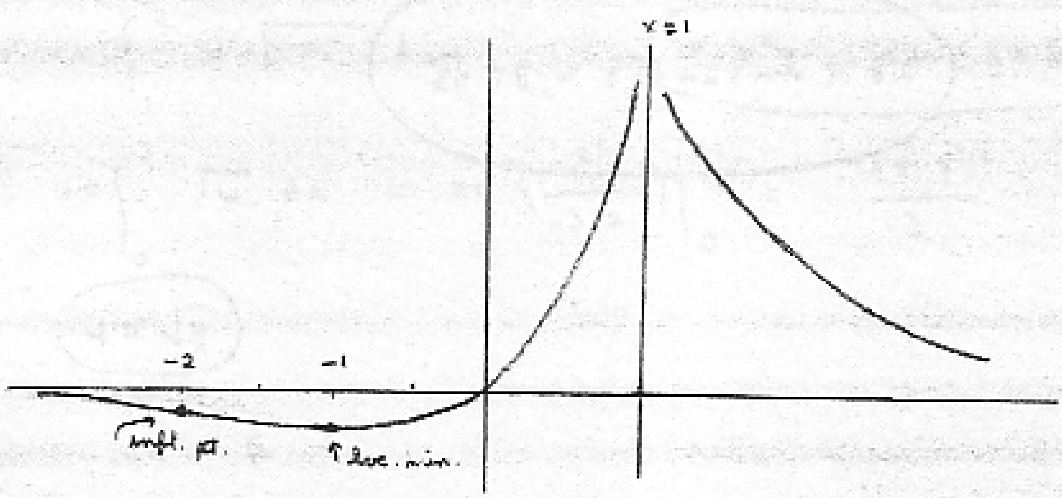
$-2 < x < 1$: $f'' > 0$, f is concave up

$1 < x < \infty$: $f'' > 0$, f is concave up

Inflection point: $(-2, -\frac{2}{9})$

(c) The line $x = 1$ is a vertical asymptote : $\lim_{x \to 1} f(x) = \infty$.

The x-axis is a horizontal asymptote : $\lim_{x \to \pm\infty} f(x) = 0$.



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9. (30) Ship A is headed east at 30 miles per hour. Ship B is headed north at 30 miles per hour. At noon ship B is 30 miles due south of ship A.

(a) What is the relative position of the ships when the distance between them is smallest?

(b) At what rate is the distance between the ships increasing at 3 p.m.?

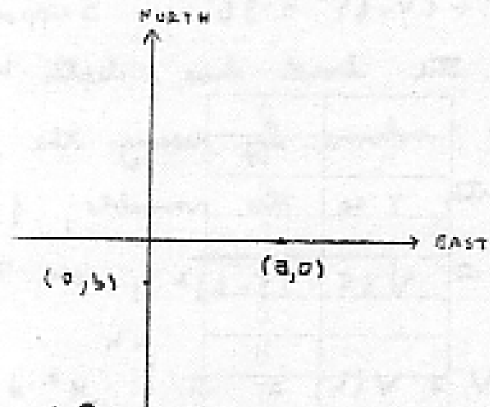
Set up coordinates as in the sketch (origin = A's position at noon).

$(a,0)$ = position of A, $(0,b)$ = position of B

$a = a(t) = 30t$

$b = b(t) = 30t - 30$

(t in hours, $t=0$ corresponds to noon)



(a) Let $s = s(t)$ be the distance between A and B.

Then $s^2 = a^2 + b^2 = (30t)^2 + (30t-30)^2 = 900\left[t^2 + (t-1)^2\right]$.

Thus $\dfrac{d(s^2)}{dt} = 900\left[2t + 2(t-1)\right] = 1800(2t-1)$

We see that s^2 decreases until $t = \frac{1}{2}$ and increases thereafter. The ships are closest when $t = \frac{1}{2}$, at which point A is 15 miles east of its position at noon and B is 15 miles south of A's position at noon.

(b) Since $\dfrac{d(s^2)}{dt} = 2s\dfrac{ds}{dt}$, the expression above for $\dfrac{d(s^2)}{dt}$ gives

$$\frac{ds}{dt} = \frac{900(2t-1)}{s} = \frac{30(2t-1)}{\sqrt{t^2 + (t-1)^2}}$$

Thus

$$\left.\frac{ds}{dt}\right|_{t=3} = \frac{30(6-1)}{\sqrt{3^2+2^2}} = \frac{150}{\sqrt{13}} \text{ mph}$$

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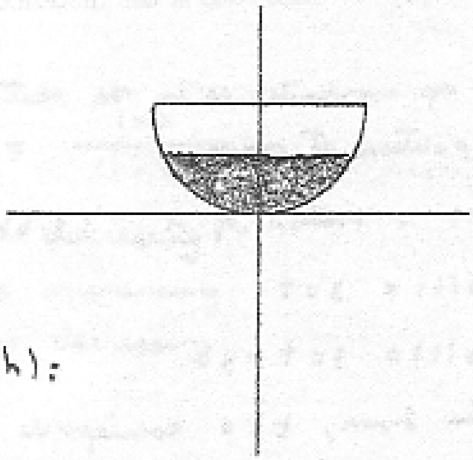
10. (20) A hemispherical bowl had radius 6 inches at its top. Water is dripping into the bowl at a rate of 2 cubic inches per minute. At what rate is the water level rising when the water is 4 inches deep?

You can get the bowl by revolving about the y-axis the lower half of the interior of the circle $x^2+(y-6)^2=36$. Suppose the water in the bowl has depth h. We get its volume by using the disk method, with y as the variable, for the function $x=\sqrt{36-(y-6)^2}$, on the interval $(0,h)$:



$$V = V(h) = \pi\int_0^h x^2\,dy = \pi\int_0^h \left(36-(y-6)^2\right)dy$$

$$= \pi\int_0^h (12y-y^2)\,dy.$$

Therefore

$$\frac{dV}{dh} = \pi(12h-h^2),$$

and

$$\frac{dV}{dt} = \pi(12h-h^2)\frac{dh}{dt}.$$

We are given that $\frac{dV}{dt} = 2\ \text{in}^3/\text{min}$, so

$$\frac{dh}{dt} = \frac{2}{\pi(12h-h^2)}.$$

Hence

$$\left.\frac{dh}{dt}\right|_{h=4} = \frac{2}{\pi(48-16)} = \frac{1}{16\pi}\ \text{in}/\text{min}.$$