ME 132, Fall 2003, Final Exam

Name:

# 1	# 2	# 3	# 4	# 5	# 6	# 7	# 8	# 9	# 10	TOTAL
18	21	15	15	27	15	15	15	21	18	180

Facts:

1. **3rd order stability test:** The roots of the third-order polynomial

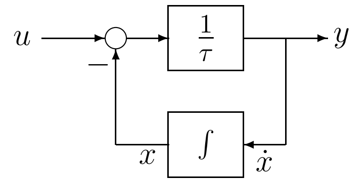
$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

   all have negative real-parts if and only if $a_1 > 0, a_3 > 0$ and $a_1 a_2 > a_3$.

2. The characteristic polynomial of the 1st order (vector) differential equation $\dot{x}(t) = Ax(t)$ is $\det(sI - A)$.

3. The characteristic polynomial of the 1st order (vector) differential equation $\dot{x}(t) = Ax(t) + Bu(t), y(t) = Cx(t) + Du(t)$ is $\det(sI - A)$.

1. The block diagram below is often called an "approximate differentiator." Note that nowhere in the block diagram is there a differentiating element.



(a) Based on the block diagram, write the differential equation relating x, $\dot{x}$ and u.

(b) Write the equation expressing y in terms of x and u

(c) Show that the transfer function from U to Y is $\frac{s}{\tau s+1}$.

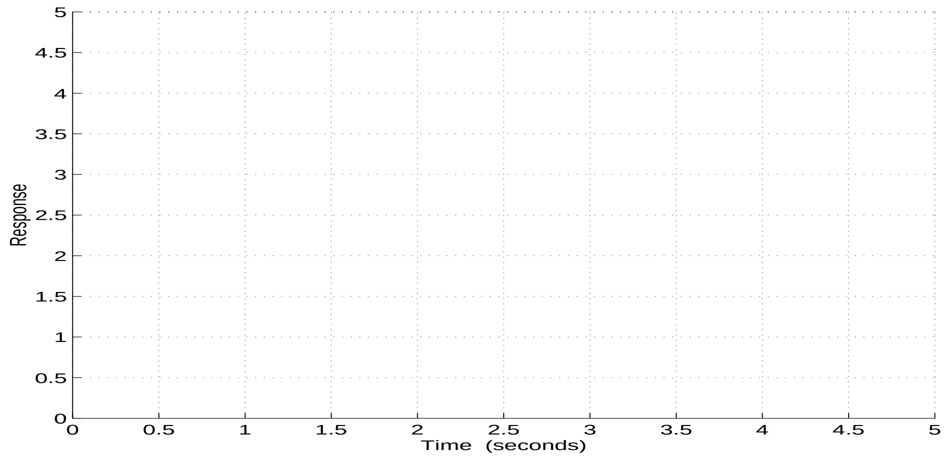
(d) Suppose that the initial condition is $x(0) = 0$. Apply a step input at $t = 0$, so $u(t) = \bar{u}$ for $t > 0$ (here, $\bar{u}$ is just some constant value). Compute the response $x(t)$, for $t \geq 0$.

(e) With $x(t)$ computed above, compute the output $y(t)$, and sketch below.

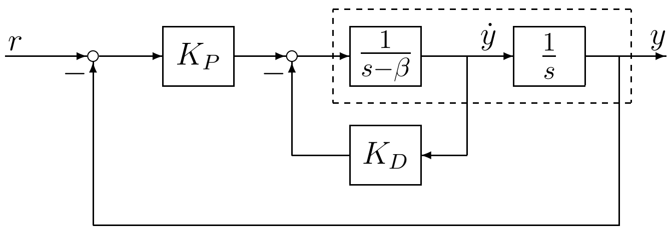
(f) Suppose that the initial condition is $x(0) = 0$, let $\tau = 0.2$. Apply a ramp input (with slope 3)

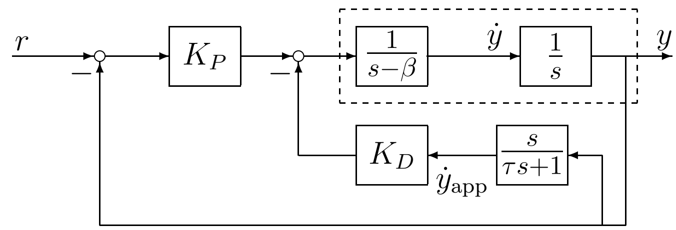
$$u(t) = 3t \quad \text{for } t \geq 0.$$

Compute the response $y(t)$, and plot. If you cannot derive the expression for y, guess what it should look like, and plot it below.



2. Block diagrams for two systems are shown below. Two of the blocks are just gains, (K_P and K_D) and the other blocks are described by their transfer functions. The constant β is positive, $\beta > 0$. The system on the left is stable if and only if $K_P > 0$ and $K_D > \beta$ (no need to check this – it is correct). What are the conditions on K_P, K_D and τ, such that the system on the right is stable? **Hint:** Note that τ is the time-constant of the filter in the approximate differentiation used to obtain $\dot{y}_{\text{app}}$ from y. The stability requirements will impose some relationship between it's cutoff frequency $\frac{1}{\tau}$ and the severity (eg., speed) of the unstable dynamics of the process, namely β.



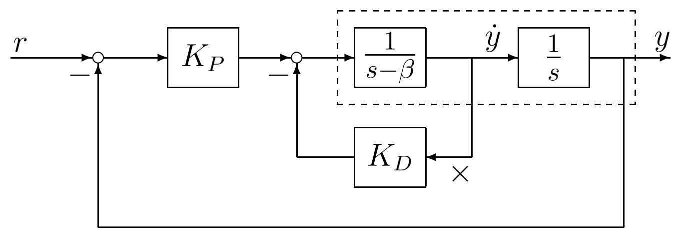


3. In computing gain and time-delay margins, we solve equations of the form

$$-1 = \gamma L(j\omega) \qquad -1 = e^{-j\omega T} L(j\omega)$$

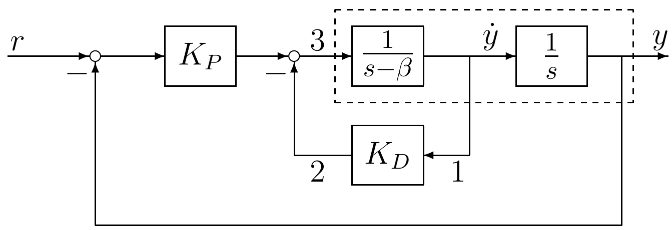
using an appropriate L, depending on the system under consideration.

For the system below, what is the appropriate L in order to compute gain and time-delay margins at the point marked by $\times$.



4. (a) For the system below, are the gain and time-delay margins at the point marked by 1 the same as the gain and time-delay margins at the point marked by 2? Justify your answer.

(b) For the system below, are the gain and time-delay margins at the point marked by 2 the same as the gain and time-delay margins at the point marked by 3? Justify your answer.



5. A hoop (of radius R) is mounted vertically, and rotates at a constant angular velocity Ω. A bead of mass m slides along the hoop, and θ is the angle that locates the bead location. $\theta = 0$ corresponds to the bead at the bottom of the hoop, while $\theta = \pi$ corresponds to the top of the hoop.

The nonlinear, 2nd order equation (from Newton's law) governing the bead's motion is

$$mR\ddot{\theta} + mg\sin\theta + \alpha\dot{\theta} - m\Omega^2 R\sin\theta\cos\theta = 0$$

All of the parameters m, R, g, α are positive.

(a) Let $x_1(t) := \theta(t)$ and $x_2(t) := \dot{\theta}(t)$. Write the 2nd order nonlinear differential equation in the state-space form

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), x_2(t)) \\ \dot{x}_2(t) &= f_2(x_1(t), x_2(t)) \end{aligned}$$

(b) Show that $\bar{x}_1 = 0, \bar{x}_2 = 0$ is an equilibrium point of the system.

(c) Find the linearized system

$$\dot{\delta}_x(t) = A\delta_x(t)$$

which governs small deviations away from the equilibrium point $(0, 0)$.

(d) Under what conditions (on m, R, Ω, g) is the linearized system stable?

(e) Show that $\bar{x}_1 = \pi, \bar{x}_2 = 0$ is an equilibrium point of the system.

(f) Find the linearized system $\dot{\delta}_x(t) = A\delta_x(t)$ which governs small deviations away from the equilibrium point $(\pi, 0)$.

(g) Under what conditions is the linearized system stable?

(h) It would seem that if the hoop is indeed rotating (with angular velocity Ω) then there would other equilibrium point (with $0 < \theta < \pi/2$). Do such equilibrium points exist in the system? Be very careful, and please explain your answer.

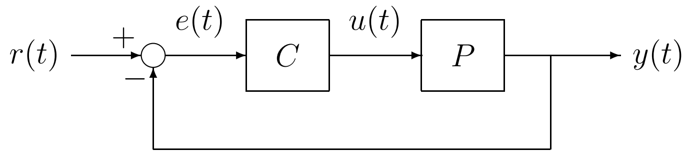
(i) Find the linearized system $\dot{\delta}_x(t) = A\delta_x(t)$ which governs small deviations away from this equilibrium point.

(j) Under what conditions is the linearized system stable?

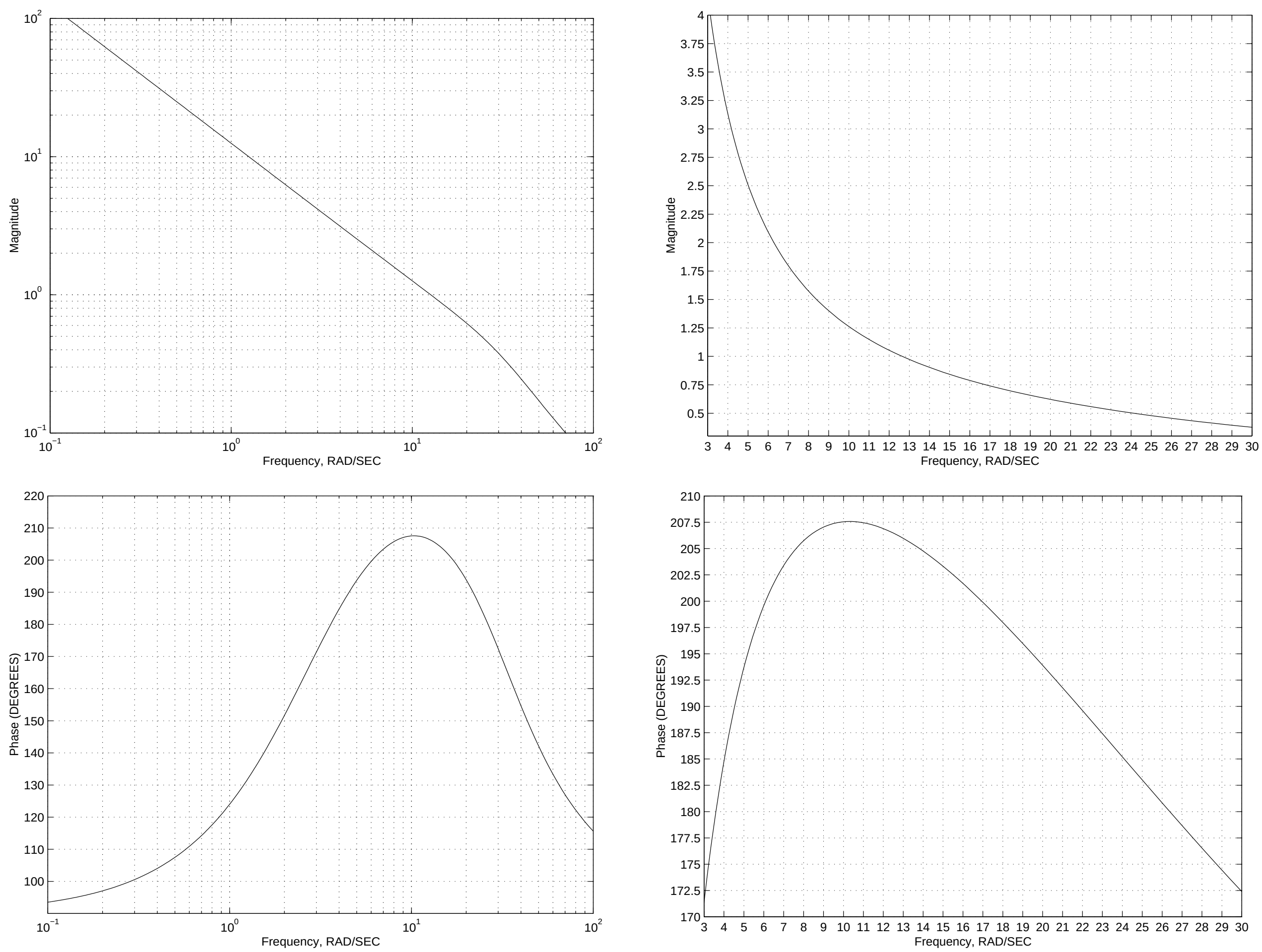
6.

7.

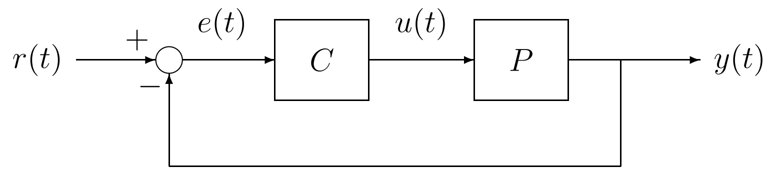
8. A closed-loop feedback system consisting of plant P and controller C is shown below.



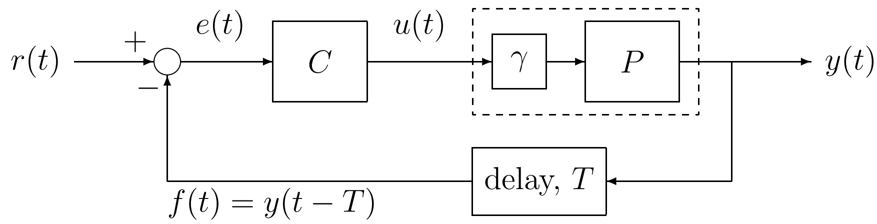
In this problem, it is known that the nominal closed-loop system is stable. The plots below are the magnitude and phase of the product $\hat{P}(j\omega)\hat{C}(j\omega)$, given both in linear and log scales, depending on which is easier for you to read. Use these graphs to compute the **time-delay margin** and the **gain margin**. Clearly indicate the gain-crossover and phase-crossover frequencies which you determine in these calculations.



9. A closed-loop feedback system consisting of plant P and controller C is shown below.

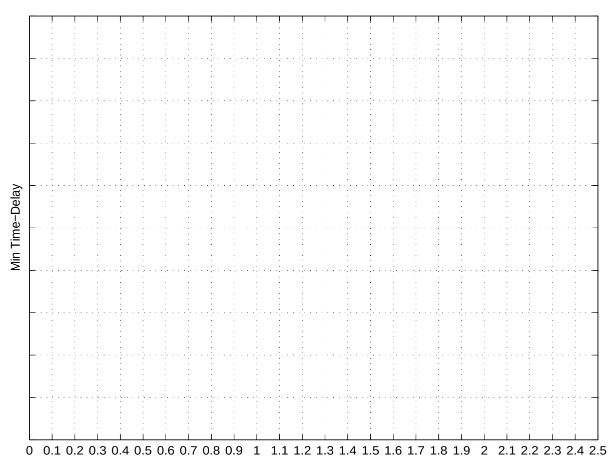


It is known that the nominal closed-loop system is stable. In the presence of gain-variations in P and time-delay in the feedback path, the closed-loop system changes to

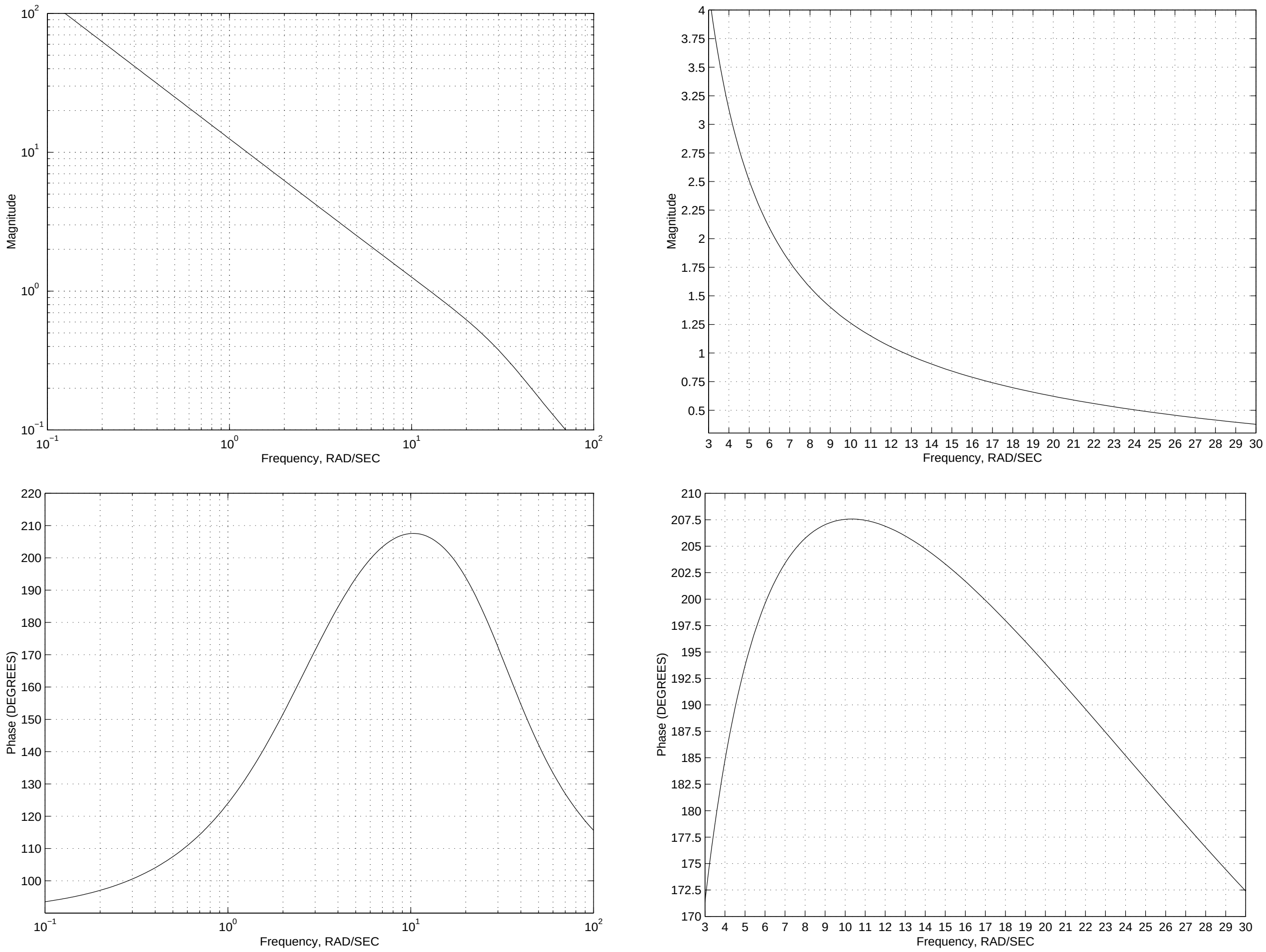


In this particular system, there is both an upper and lower gain margin - that is, **for no time-delay**, if the gain γ is decreased from 1, the closed-loop system becomes unstable at some (still positive) value of γ; and, if the gain γ is increased from 1, the closed-loop system becomes unstable at some value of $\gamma > 1$. Let γ_l and γ_u denote these two values, so $0 < \gamma_l < 1 < \gamma_u$.

For each fixed value of γ satisfying $\gamma_l < \gamma < \gamma_u$ the closed-loop system is stable. For each such fixed γ, compute the minimum time-delay that would cause instability. Specifically, do this for several (say 8-10) γ values satisfying $\gamma_l < \gamma < \gamma_u$, and plot below.



The data on the next two pages are the magnitude and phase of the product $\hat{P}(j\omega)\hat{C}(j\omega)$. They are given in both linear and log spacing, depending on which is easier for you to read. Use these graphs to compute the time-delay margin at many fixed values of γ satisfying $\gamma_l < \gamma < \gamma_u$.



Magnitude
Frequency, RAD/SEC
Magnitude
Frequency, RAD/SEC
Phase (DEGREES)
Frequency, RAD/SEC
Phase (DEGREES)
Frequency, RAD/SEC

10. A popular recipe from the 1940's for designing PID controllers (PI control, with inner-loop rate-feedback) is the Ziegler-Nichols method. It is based on simple experiments with the actual process, not requiring ODE models of the process. Nevertheless, we can analyze the method on specific process transfer functions.

    There are two Ziegler-Nichols versions. The first version of the method is as follows:

    Ziegler-Nichols PID Design Method

    Step 1: Connect plant in negative feedback with a proportional gain controller

    Step 2: Slowly increase gain of proportional controller. At some value of gain, the closed-loop system will become unstable, and start freely oscillating. Denote the value of this critical proportional gain as K_c and the period of the oscillations at T_c.

    Step 3: For the actual closed-loop system, use gains as below

$$K_P = 0.6K_c, \qquad K_I = 1.2\frac{K_c}{T_c}, \qquad K_D = \frac{3}{40}K_cT_c$$

    (a) Suppose the plant has transfer function

$$G(s) = \frac{1}{s(\tau s + 1)}$$

    where τ is a fixed, positive number. What difficulties arise in attempting to use the Zeigler-Nichols design method?

(b) Suppose the plant has transfer function

$$G(s) = \frac{-\tau_1 s + 1}{s(\tau_2 s + 1)}$$

where τ_1 and τ_2 are some fixed, positive numbers. Imagine that you carry out the Step 1 & 2 of the procedure directly on the plant. What will the parameters K_c and T_c be equal to? Your answers should be exclusively in terms of τ_1 and τ_2.