CS270 Midterm Exam
Due: Wednesday,March 21, 2007

1. Given an array A of n distinct integers, let $a_{(k)}$ denote the k-th-smallest element. Let α be a positive number less than 1. The α-approximate sorting problem is to rearrange the elements into an array B of length n such that, for all k, the position of $a_{(k)}$ in B lies between $k(1-\alpha)$ and $k(1+\alpha)$. Give a linear-time algorithm to solve this problem.

2. You are given a connected graph G with n vertices, m edges and two distinguished vertices s and t. Each edge is labeled with a positive integer $t(u,v)$ representing the time at which the edge disappears from the graph. The graph is presented as an array of records, where each record is of the form $(u, v, t(u,v))$. These records are arranged in increasing order of $t(u,v)$. Let D denote the earliest time at which, because of the disappearance of edges, the graph no longer contains a path between s and t. Give an algorithm for computing D with running time $o(m \log m)$. Note: this is little-o, not big-O. Hint: make use of the union-find data structure.

3. *Local Alignment* This is a variation on the string alignment problem discussed in class. The score of an alignment of two sequences is $AM - BS - CI$, where A, B and C are positive constants, M is the number of matches, S is the number of substitutions, and I is the number of insertions and deletions. Given symbol strings A and B of lengths m and n, you wish to compute the highest possible score of an alignment of a substring of A with a substring of B, where "substring" means "sequence of consecutive elements." In other words, the algorithm is free to ignore some prefix and some suffix of each of the given strings and align what remains. Give an algorithm to solve this problem in time $O(mn)$.

4. In the *fractional vertex cover problem* you are given a graph $G = (V, E)$ with a positive numerical weight $a(v)$ on each vertex v. You are required to assign a nonnegative value $x(v)$ to each vertex so as to minimize $\sum_{v \in V} a(v)x(v)$ subject to the constraint that, for each edge (u, v), $x(u) + x(v) \geq 1$.

   (a) Express this problem as a linear programming problem and describe the dual problem.

   (b) The *integer vertex cover problem* is the same as the above problem, except that, for each vertex v, $u(v)$ must be 0 or 1. Show that, given an optimal solution to the fractional vertex cover problem, one can efficiently derive a solution to the integer vertex cover problem whose cost $\sum_{v \in V} a(v)x(v)$ is at most twice the cost of an optimal solution to the integer vertex cover problem.

5. Consider a version of the augmenting-path algorithm for the max-flow problem in which, at each iteration, an augmenting path with the largest possible bottleneck capacity is chosen. Prove that, if the capacities are integers, then the number of iterations is bounded above by a polynomial function of n, the number of vertices, m, the number of edges, and $\log D$, where D is the largest edge capacity. Hint: By considering $s - t$ cuts, estimate the ratio by which the gap between the current flow value and the maximum flow value is reduced at each iteration.

6. You are designing a laboratory. There are m experiments $X_1, X_2, \cdots, X_m$ that you would like the laboratory to perform, and n machines $Y_1, X_2, \cdots Y_n$ that you can purchase. For each experiment X_i there is a set $S_i \subseteq \{Y_1, X_2, \cdots Y_n\}$ such that experiment X_i can be performed if and only if you purchase all the machines in S_i. Machine Y_j costs a_j, and the ability to perform experiment X_i has value b_i. You must decide which machines to purchase, so as to maximize the sum of the values of the experiments that can be performed, minus the sum of the costs of the machines that are purchased. Show that this problem can be solved by computing a minimum-capacity $s - t$ cut in a suitably defined flow network. Hint: the minimum $s - t$ cut will contain an edge corresponding to each experiment that cannot be performed, and an edge corresponding to each machine that is purchased.