# Fall 2009 - Lee - Midterm 2 solutions

## Problem 1 Solutions

**Part A**

Because the middle slab is a conductor, the electric field inside of the slab must be 0.

**Parts B and C**

Recall that to find the electric field near a plane charge of charge density $\sigma$, you can draw a Gaussian pillbox through it. Because the electric field points perpendicular to the plane, we only have a flux through the ends of the box. (See example 22-7 in the book, page 598)

$$\oint \overrightarrow{E} \cdot \overrightarrow{dA} = 2EA = \frac{\sigma A}{\epsilon_0} E = \frac{\sigma}{2\epsilon_0}$$

The first equality in the first equation is because the electric field is perpendicular and uniform to the ends of the box, and the second is Gauss's law. (Note that $A$ here is the area of the box, not of the plates)

Because we have symmetry, all charge distributions will be sheet charges. If we define the positive direction to the right, and watch our sign conventions, $E_l$ (the electric field between the left plate and the middle slab) and $E_r$ (the electric field between the middle slab and the right plate) are:

$$E_l = \frac{\sigma}{2\epsilon_0} - \frac{\sigma_l}{2\epsilon_0} - \frac{\sigma_r}{2\epsilon_0} - \frac{-\sigma}{2\epsilon_0}$$
$$E_r = \frac{\sigma}{2\epsilon_0} + \frac{\sigma_l}{2\epsilon_0} + \frac{\sigma_r}{2\epsilon_0} - \frac{-\sigma}{2\epsilon_0}$$

Where $\sigma_l$ and $\sigma_r$ are the charge densities on the left and right side of the conducting slab. Since the slab is neutral, these two charge densities are equal and opposite, so the middle two terms will add to 0 in both cases. Thus:

$$E_l = E_r = \frac{\sigma}{\epsilon_0}$$

**Parts D and E**

Now we draw a pillbox with one end in the space between the slab and plate, and one end in the middle of the slab. Since the electric field inside the conductor is 0:

$$\oint \overrightarrow{E} \cdot \overrightarrow{dA} = -EA = \frac{\sigma_l}{\epsilon_0}$$
$$\oint \overrightarrow{E} \cdot \overrightarrow{dA} = EA = \frac{\sigma_r}{\epsilon_0}$$

Note that the flux is negative if the box contains the left surface of the slab, and is positive if the box contains the right surface of the slab. Then, using our result from parts B and C for electric field, we find

$$\sigma_l = -\sigma$$
$$\sigma_r = \sigma$$

# Problem 2

**Problem:**
Suppose in the figure that $C_1 = C_2 = C_3 = C$ and $C_4 = 2C$. The charge on $C_2$ is $Q$.
**a)** Determine the charge on each of the other capacitors and the voltage across each capacitor.
**b)** Determine the voltage $V_{ab}$ across the entire combination.

**Solution:**
**a)** Since the capacitors $C_1$ and $C_2$ are in series, the charges on each must be equal:

$$Q_1 = Q_2 = Q \qquad \text{(3 points)}$$

To find the charge on $C_3$, use the fact that since $C_1$ and $C_2$ are in parallel with $C_1$, the voltage across $C_1$ and $C_2$ must be equal to the voltage across $C_3$:

$$V_3 = V_1 + V_2$$
$$\Rightarrow \frac{Q_3}{C_3} = \frac{Q_1}{C_2} + \frac{Q_1}{C_1} \Rightarrow Q_3 = 2Q \qquad \text{(3 points)}$$

To find $Q_4$, note that since the negative plate of $C_4$ is connected to the positive plates of $C_1$ and $C_3$, their charges must be equal:

$$Q_4 = Q_1 + Q_3 = 3Q \qquad \text{(2 points)}$$

To find the voltages, use the equation $Q = CV$:

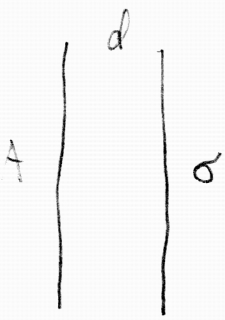
$$V_1 = V_2 = \frac{Q}{C}$$
$$V_3 = \frac{2Q}{C}$$
$$V_4 = \frac{3Q}{2C} \qquad \text{(2 points)}$$

**b)** To find the $V_{ab}$, choose any path from $a$ to $b$ and add the voltages!

$$V_{ab} = V_4 + V_3 = V_4 + V_1 + V_2 \qquad \text{(5 points)}$$

$$\Rightarrow V_{ab} = \frac{Q_4}{C_4} + \frac{Q_3}{C_3} = \frac{7}{2}\frac{Q}{C} \qquad \text{(5 points)}$$

3. (a)




$$\begin{cases} Q = \sigma A \ (2') \\ V = Ed \ (2') \\ E = \dfrac{\sigma}{\varepsilon_0} \ (2') \\ C = \dfrac{Q}{V} \ (2') \end{cases} \Rightarrow C = \frac{\varepsilon_0 A}{d} \ (2')$$

(b) $V = V_0$ (The battery holds V constant) (1')

$C = kC_0$ (1')

$Q = CV = kC_0V_0 = kQ_0$ (1')

$\text{Energy} = \frac{1}{2}CV^2 = \frac{1}{2}kC_0V_0^2 = k\,\text{Energy}_0$ (2')

(c) $Q = Q_0$ (Charge conservation) (1') $C = \frac{C_0}{k}$ (1')

$V = \frac{Q}{C} = k\frac{Q_0}{C_0} = kV_0$ (1')

$\text{Energy} = \frac{Q^2}{2C} = \frac{kQ_0^2}{2C_0} = k\,\text{Energy}_0$ (1')

The increased energy comes from the work done by the external source when it pulls out the dielectric. (1')

(WR stands for wrong reasoning)

4. a) $V = \dfrac{Q}{4\pi\varepsilon_0 D} \times 2$

$= \dfrac{Q}{2\pi\varepsilon_0 D}$ (5')

b) $\begin{cases} V = \dfrac{Q}{4\pi\varepsilon_0 r} \times 2 & (5') \\ r = \sqrt{D^2+y^2} & (2') \end{cases} \Rightarrow V = \dfrac{Q}{2\pi\varepsilon_0 \sqrt{D^2+y^2}}$ (3')

c) From symmetry, we know $E_x = E_z = 0$ (3')

$E_y = -\dfrac{\partial V}{\partial y}$ (3')

$= \dfrac{Q\,y}{2\pi\varepsilon_0 (D^2+y^2)^{3/2}}$ (4')

P.S. I need to see the symmetry argument that yields $E_x = E_z = 0$. You can't do $\frac{\partial V}{\partial x}$ or $\frac{\partial V}{\partial z}$ because you haven't calculated V as a function of x or z

(WR stands for wrong reasoning.)

# Professor Lee Physics 7B Midterm 2

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## Problem 5 - 25 points

*A solid spherical ball that is uniformly charged with a charge density $\rho$ and total charge $Q$ has a radius $R$. Take the zero point of the potential to be at infinity as usual.*

*(a) [10 Points] What is the potential a distance of $2R$ from the ball's center? Derive the answer here from a line integral approach, do not just state a formula*

We need to find the potential using the "line integral approach", *i.e.* find the potential from the electric field:

$$V = -\int \vec{E} \cdot d\vec{l}$$

To find the potential, we need to find the electric field. This can be done using Gauss' Law. Because we are asked for the potential at a point $2R$ *outside* the sphere, we only need to find the electric field for points outside the sphere (*i.e.* for $r > R$). Exploiting the spherical symmetry of the problem, we choose a spherical Gaussian surface of radius $r > R$:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$
$$\rightarrow Q_{\text{enc}} = +Q \text{ for this surface} \rightarrow$$
$$\left|\vec{E}\right| (\text{SA})_{\text{Gaussian Sphere}} = \frac{Q}{\epsilon_0}$$
$$\left|\vec{E}\right| 4\pi r^2 = \frac{Q}{\epsilon_0}$$

so

$$\vec{E} = k\frac{Q}{r^2}\hat{r} \text{ for } r > R$$

Of course we could have just written this down since outside the sphere it looks like a point charge. Getting the electric field for $r > R$ using either Gauss' law or just writing it down is worth 4 points.

We can find the potential using the line integral approach. For a point a distance $r$ ($r > R$) from the origin, the potential is given by (taking the potential to be zero at infinity):

$$V(r) - \underbrace{V(\infty)}_{0} = V(r) = -\int_{\infty}^{r} \vec{E}_{(r>R)} \cdot d\vec{l}$$

Evaluating this:

$$\begin{aligned} V(r) &= -\int_{\infty}^{r} \vec{E}_{(r>R)} \cdot d\vec{l} \\ &= -\int_{\infty}^{r} k\frac{Q}{r'^2} dr' \\ &= k\frac{Q}{r} \end{aligned}$$

So that at a point $2R$ from the balls center, the potential is given by:

$$\boxed{V(2R) = k\frac{Q}{2R}}$$

Point breakdown:

1. Electric field outside the sphere (w/ or w/o Gauss' law): 4 pts
2. Line integral for the potential: 2 pts
3. Correct limits on line integral: 2 pts
4. Using $\vec{E}$ for $r > R$ in integral: 1 pt
5. Getting the correct answer: 1 pt

*(b) [5 Points] A tiny test charge with mass $m$ and charge $q$ is released just at the sphere's surface at $R$. How fast is it going by the time it reaches $2R$?*

We use energy conservation to solve this one:

$$E_i = E_f \rightarrow \frac{1}{2}mv_i^2 + U(R) = \frac{1}{2}mv_f^2 + U(2R)$$

The charge is released from rest so $v_i = 0$. The potential energy of the charge is given by $U(r) = qV(r)$. Plugging these into the above we get the answer:

$$\begin{aligned}
v_f^2 &= \frac{2q}{m}(V(R) - V(2R)) \\
&= \frac{2q}{m}\left(-\int_{2R}^{R} k\frac{Q}{r'^2}dr'\right) \\
&= \frac{2kqQ}{m}\left(\frac{1}{R} - \frac{1}{2R}\right) \\
&= \frac{kqQ}{mR}
\end{aligned}$$

Point breakdown:

1. Energy conservation and $U = qV$: 2 pts
2. $\Delta V = V(2R) - V(R)$ (can calculate w/ line integral or using $V = kQ/r$ from part a) : 2 pts
3. Correct answer: 1 pt

*(c) [10 Points] What is the potential inside the ball as a function of radius $r$? Does it rise or fall with radius? What is the potential at the center of the ball?*

For $r < R$ the potential is given by:

$$V(r) = V(R) - \int_{R}^{r} \vec{E}_{(r<R)} \cdot d\vec{l}$$

where $V(R) = kQ/R$. We need to find the electric field inside the sphere to calculate this. We will use Gauss' law with a spherical Gaussian surface of radius $r < R$. The charge enclosed by this surface is:

$$Q_{\text{enc}} = \rho\frac{4\pi}{3}r^3 = \frac{Q}{\frac{4\pi}{3}R^3}\frac{4\pi}{3}r^3 = Q\frac{r^3}{R^3}$$

So using Gauss' law:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$
$$\left|\vec{E}\right| 4\pi r^2 = \frac{Q}{\epsilon_0}\frac{r^3}{R^3}$$
$$\rightarrow \vec{E} = kQ\frac{r}{R^3}\hat{r}$$

Using the above we have:

$$-\int_R^r \vec{E}_{(r<R)} \cdot d\vec{l} = -\int_R^r \frac{kQ}{R^3} r dr$$
$$= \frac{kQ}{2R^3}(R^2 - r^2)$$

So that the potential at radius $r$ is given by:

$$V = V(R) - \int_R^r \vec{E}_{(r<R)} \cdot d\vec{l}$$

$$\boxed{V(r) = kQ\left(\frac{1}{R} + \frac{1}{2R} - \frac{r^2}{2R^3}\right)}$$

From the above it is clear that the potential decreases with increasing radius. Physically, this is because it takes energy to bring a test charge at the surface of the sphere to a point inside the sphere.
The potential at the center of the sphere $r = 0$ is given by:

$$V(r=0) = \frac{3}{2}\frac{kQ}{R} = \frac{3}{2}V(R)$$

Note how the potential is larger at the center of the sphere than at the surface, as we discussed above.

Point breakdown:

1. Electric field for $r < R$: 3 pts
2. $V = -\int \vec{E} \cdot d\vec{l}$: 1 pt
3. Adding potential from $\infty \rightarrow R$/ correct limits on above: 2pts
4. Correct answer for $V(r)$: 1 pt
5. Discussing that potential decreases w/ radius: 2 pts (1 pt for saying it decreases, 1 pt for giving a reason why)
6. Potential at the center of the ball: 1 pt