1. True or False (15 points total)

(a) The cross product of two UNIT vectors has length 1.

(b) The graph of the curve $r = 4\cos 2\theta$ is contained inside a circle of radius 2.

$r = 4$

(c) If $\vec{u}$ and $\vec{v}$ are nonparallel vectors in $\mathbf{R}^3$ then up to multiplication by a scal
$u \times v$ is the ONLY vector perpendicular to both u and v.

T

(d) The function $f(x, y) = \sin xy + x^2$ is an example of a function $f : \mathbf{R}^1 \to \mathbf{R}^2$.

F $\qquad \mathbb{R}^2 \to \mathbb{R}$

(e) If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ then $\vec{b} = \vec{c}$. (Here we are taking the dot product of vectors in $\mathbf{R}^3$.)

F $\to$ sums add up to the same only

$\vec{a} = \langle 1, 1, 1 \rangle$

$\vec{b} = \langle 3, 1, 2 \rangle$

$\vec{c} = \langle 2, 3, 1 \rangle$.

2. Multiple Choice and Short Answer (15 points total)

I If u and v are unit vectors, what can be said about $|u \cdot v|$?

A. It is always equal to 1

B. It is always ≤ 1

C. It is always > 1

D. None of the above.

II Let $f(z, x)$ be a function of two variables. Which of the following represents the partial derivative with respect to the variable x.

A. $\lim_{h \to 0} \frac{f(z+h, x) - f(z, x)}{h}$

B. $\lim_{h \to 0} \frac{f(z, x+h) - f(z, x)}{h}$

C. $\lim_{h \to 0} \frac{f(x+h, z) - f(x, z)}{h}$

D. $\lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$

III The graph of the parametric curve defined by

$$r(t) = (\cos 2t, e^{\sin t}, \sin 2t)$$

A. ~~Lies on a sphere~~

B. Lies on a cylinder

C. Lies on a cylindrical cone

D. Lies to the police.

IV A parametric curve in $\mathbf{R}^3$ is best described by

A. a function $f : \mathbf{R}^3 \to \mathbf{R}^1$

B. a function $f : \mathbf{R}^1 \to \mathbf{R}^2$

C. a function $f : \mathbf{R}^2 \to \mathbf{R}^3$

D. a function $f : \mathbf{R}^1 \to \mathbf{R}^3$

$f(t) = \langle t, t^2, t^3 \rangle$

(5 points)

(a) Let $\vec{u} = (2, 0, 1)$ and $\vec{v} = (4, 5, 3)$. Compute the vector projection of $\vec{u}$ onto $\vec{v}$.

$= \frac{11}{50}\langle 4, 5, 3\rangle$

$= \boxed{\left\langle \frac{44}{50}, \frac{55}{50}, \frac{33}{50} \right\rangle}$

$2 + 0 + 3$

(b) What would the projection be if I had given you two perpendicular vectors?

0. $\quad \vec{u} \cdot \vec{v} = |u||v|\cos\theta, \theta = 90°$

(c) Find a vector perpendicular to $\vec{u}$ but NOT perpendicular to $\vec{v}$.

$\langle 2, 0, 1\rangle \cdot \langle a, b, c\rangle = 0$

$2a + c = 0$

$2a = -c \quad a = 1, c = -2$

$\langle 1, 0, -2\rangle \cdot \langle 4, 5, 3\rangle = 4 - 6 = -2 \rightarrow$ not $\perp$

$\boxed{\langle 1, 0, -2\rangle}$

4. (20 points total) Consider the function

$$F(x, y) = x\cos(\pi y) + x^2 y$$

and its graph $z = f(x, y)$.

(a) Compute the partial derivatives F_x and F_y.

$F_x = \cos(\pi y) + 2xy$

$F_y = x\pi(-\sin(\pi y)) + x^2 = -\pi x \sin(\pi y) + x^2.$

(b) Compute the equation of the plane tangent to the surface at $(1, 1, 0)$.

$F_x(1,1,0) = \cos(\pi) + 2 = -1 + 2 = 1.$

$F_y(1,1,0) = -\pi\sin(\pi) + 1 = 0 + 1 = 1.$

$z - \cancel{z_0}^{0} = F_x(1,1,0)(x-1) + F_y(1,1,0)(y-1) \rightarrow z = (x-1) + (y-1) \rightarrow \boxed{0 = x + y - z - 2}$

(c) Let C be the curve which is the intersection of this surface and the plane $y = 1$. Write down parametric equations for C.

$y = 1$
$x = t$
$z = t - 1$

$0 = x + 1 - z - 2$
$0 = x - z - 1.$
$0 = t - z - 1.$
$z = t - 1$

$y = 1.$
$x = t.$
$z = t^2 - t$

$z = x\cos(\pi) + x^2.$
$z = -x + x^2.$
$z = -t + t^2$

(d) Compute parametric equations for the line tangent to C at the point $(1, 1, 0)$. at $t = 1$.

$\frac{dy}{dt} = 0$
$\frac{dx}{dt} = 1$
$\frac{dz}{dt} = 1$

$x = 1 + t$
$y = 1$
$z = 0 + t.$

$\frac{dy}{dt} = 0$
$\frac{dx}{dt} = 1$
$\frac{dz}{dt} = 2t - 1 \rightarrow t = 1, \frac{dz}{dt} = 1.$

$\boxed{y = 1 \quad z = 0 + t \quad x = 1 + t}$

(e) Show that your line from part (d) is contained in the plane in part (b).

$0 = x + y - z - 2.$

$0 = (1 + t) + 1 - t - 2.$
$0 = 2 - 2 = 0$ ✓

$0 = (1 + t) + 1 - (t) - 2$
$= 2 - 2 = 0$ ✓

5. (a) (5 points) Suppose a differentiable function $f(x,y)$ satisfies

$$\frac{\partial f}{\partial x} = e^{-x^2}, \quad \frac{\partial f}{\partial y} = \cos y$$

And that $x = r\cos\theta$ and $y = r\sin\theta$. Calculate $\partial f/\partial r$ and $\partial f/\partial \theta$ in terms of r and θ.

$\frac{\partial u}{\partial r}$ $(e^{-x^2})(\cos\theta) + (\cos y)(\sin\theta)$

$\cos\theta + \cos(r\sin\theta)\sin\theta$

$\frac{\partial u}{\partial \theta} = (e^{-x^2})(-r\sin\theta) + (\cos y)(r\cos\theta)$

$+ \cos(r\sin\theta) r\cos\theta$.

(b) (5 points) Explain why the first part of this problem (5(a)) wouldn't have made any sense if I had said instead that

$$\frac{\partial f}{\partial x} = \sin y, \quad \frac{\partial f}{\partial y} = \cos x$$

then $f(x,y) = x\sin(y) + g(y)$

$\frac{\partial f}{\partial y} = x\cos(y)$

no way this can be changed into $\frac{\partial f}{\partial y} = \cos x$

also,

$\frac{\partial^2 f}{\partial y \partial x} = \cos y$, $\frac{\partial^2 f}{\partial x \partial y} = -\sin x$ → can't match, so by Clairaut's theorem, no such f exists

(c) (Extra Credit) If you tried to actually find the function $f(x,y)$ in part (a), I bet you failed because in fact, e^{-x^2} doesn't have an antiderivative! At least not in terms of functions that we can write down. However, it is true that

$u = \cos\theta$ $du = \sin\theta\, d\theta$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

$x = r\cos\theta$ $dx = -r\sin\theta\, d\theta$

$-r\int e^{-r^2\cos^2\theta} \sin\theta\, d\theta$

$= -\int r e^{-r^2u^2}\,du$.

For extra credit, prove this fact!

Warning: This is very hard! Only attempt if you have completed everything else. We'll actually learn how to do this later this semester on a homework problem, but if you can solve it now, all the better!)

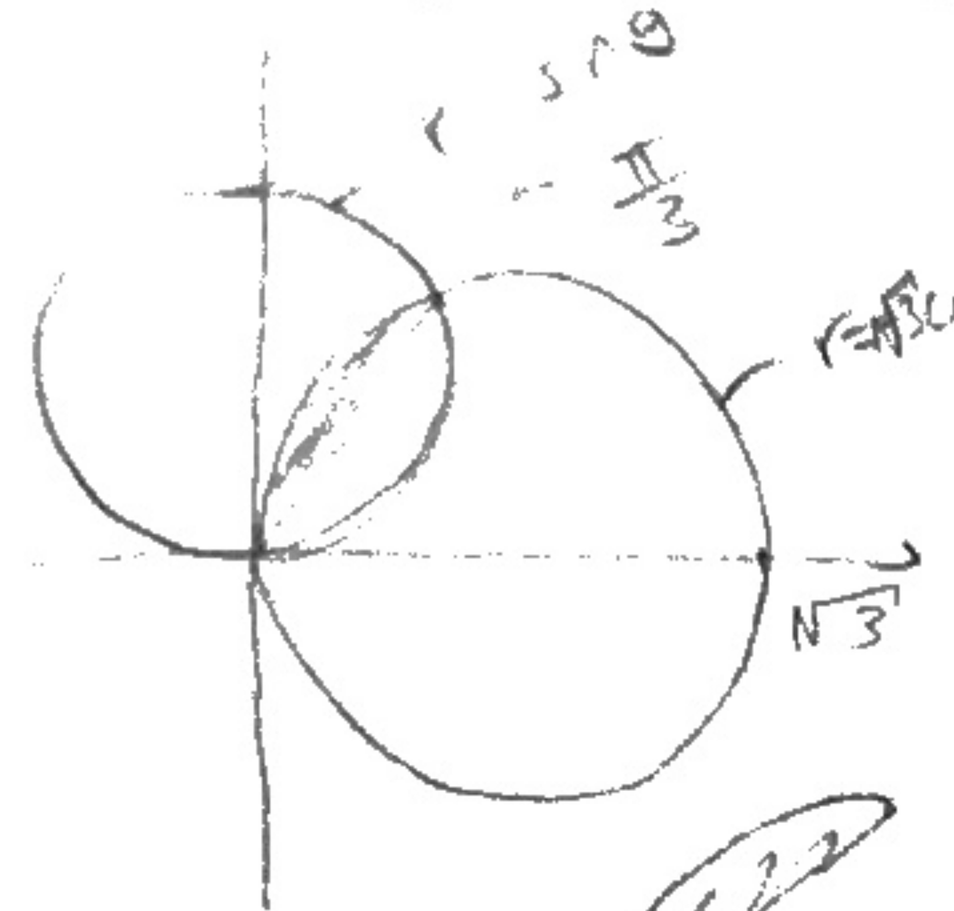
$\frac{d}{dr}\int r e^{-r^2u^2}\,du = ?$ $\int r^2 e^{-r^2u^2}\,du$

$z = r^2$ $dz = 2r\,dr$

$\int (r e^{r^2})^{u^2}\,du$ ... r, u both variable

6. (10 points) Find the area of the region that lies inside both of the following curves:

$$r = \sqrt{3}\cos\theta, \qquad r = \sin\theta.$$



$\sqrt{3}\cos\theta = \sin\theta$

$\sqrt{3} = \tan\theta,\ \theta = \frac{\pi}{3}$

$\sqrt{3}\cos\theta = 0$ at $\frac{\pi}{2}$

so $\frac{\pi}{3} \to \frac{\pi}{2}$

$$= \int_{\pi/3}^{\pi/2} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_{\pi/3}^{\pi/2} 3\cos^2\theta \, d\theta = \frac{3}{2}\int_{\pi/3}^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{3}{4}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{\pi/3}^{\pi/2} = \frac{3}{4}\left(\frac{\pi}{2} + \cancel{\frac{\sin\pi}{2}}^{\,0} - \frac{\pi}{3} - \frac{\sin(2\pi/3)}{2}\right)$$

$\frac{\pi}{8} - \frac{3\sqrt{3}}{16}$

$r = \sin\theta = 0,\ \theta = 0$

$0 \to \frac{\pi}{3}$

$$= \int_0^{\pi/3} \frac{1}{2}(\sin^2\theta)\, d\theta = \frac{1}{2}\int_0^{\pi/3} \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/3} = \frac{1}{4}\left(\frac{\pi}{3} - \frac{\sin\frac{2\pi}{3}}{2} - 0 + \frac{\sin(0)}{2}\right) = \frac{\pi}{12} - \frac{\sqrt{3}}{16}$$

$$\frac{3}{4}\left(\frac{3\pi}{6} - \frac{2\pi}{6} - \frac{\sin(2\pi/3)}{2}\right) + \frac{1}{4}\left(\frac{\pi}{3} - \frac{\sin(2\pi/3)}{2}\right)$$

$$= \frac{3}{4}\left(\frac{\pi}{6} - \frac{\sin(2\pi/3)}{2}\right) + \frac{\pi}{12} - \frac{\sin(2\pi/3)}{8}$$

$$= \frac{3\pi}{24} - \frac{3\sin(2\pi/3)}{8} + \frac{2\pi}{24} - \frac{\sin(2\pi/3)}{8}$$

$$= \frac{5\pi}{24} - \frac{\sin(2\pi/3)}{2}$$

$\sin(2\pi/3) = \sin(\pi/3) = \frac{\sqrt{3}}{2}$

$$= \boxed{\frac{5\pi}{24} - \frac{\sqrt{3}}{4}}$$

7. (10 points total)

(a) Find the intersection point of the lines defined parametrically by

$$x = 2t + 1, y = 3t + 2, z = 4t + 3,$$

and

$$x = s + 2, y = 2s + 4, z = -4s - 1.$$

$(2t+1 = s+2) \cdot 2 =$ $\quad 4t + 2 = 2s + 4$

$3t+2 = 2s+4$ $\quad - \; 3t + 2 = 2s + 4$

$t = 0$

$0 + 2 = 2s + 4$

$-2 = 2s \rightarrow s = -1$

$2(0) + 1 = -1 + 2$ ✓ $\quad x = 1 \quad \boxed{(1,2,3)}$

$1 = 1$ $\quad y = 2$

$3(0) + 2 = 2(-1) + 4$ $\quad z = 3$

$2 = 2$ ✓

$4(0) + 3 = 4 - 1$

$3 = 3$ ✓

(b) Find the equation of the plane spanned by these lines.

$\vec{r}_t = \langle 2, 3, 4 \rangle \rightarrow \langle 2,3,4 \rangle \times \langle 1,2,-4 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix}$

$\vec{r}_s = \langle 1, 2, -4 \rangle$

$= (-12 - 8)\hat{i} - (-8 - 4)\hat{j} + (4 - 3)\hat{k}$

$= -20\hat{i} + 12\hat{j} + \hat{k}$

$= -20(x-1) + 12(y-2) + 1(z-3) = 0$

$\boxed{-20x + 12y + z = 7}$

3. (10 points) Compute the following limit or show it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4} \to \text{continuous}$$

$x = y, z = 0$ $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$

$z, y = 0$ $\lim_{(x,y,z)\to(0,0,0)} \frac{z^3}{z^2+z^4} = \lim_{(x,y,z)\to(0,0,0)} \frac{z}{1+z^2} = \frac{0}{1} = 0$

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