Midterm 2
Ramesh
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Problem 1 (25 points)

A parallel plate capacitor has area $A = 20\text{cm}^2$ and a plate separation of $d = 4\text{mm}$.

(a) *If the breakdown field is $E_0 = 3 \times 10^6$V/m, calculate the maximum voltage and charge the capacitor can hold.*

The breakdown field is the minimum electric field at which a material ionizes; in air, this means a spark forms.

Maximum voltage:

$$V_{max} \quad = \quad E_0 d = 3 \times 10^6 \times 0.004 = 12\text{kV}$$

To find the maximum charge corresponding to $V = 12$kV, we need the capacitance of the parallel plate capacitor,

$$C \quad = \quad \frac{\epsilon_0 A}{d} = \frac{8.85 \times 10^{-12} \times 0.002}{0.004} = 4.425\text{pF}$$

The charge is determined from $C = Q/V$:

$$Q \quad = \quad CV = 1.2 \times 10^4 \times 4.425 \times 10^{-12} = 53.1\text{nC}$$

(b) *A Teflon sheet (dielectric constant $\kappa = 2.1$) is slid between the plates, filling the volume. Find the new capacitance and maximum charge if the breakdown field is 25 times larger than air.*

The dielectric increases the capacitance by a factor κ,

$$C_{new} \quad = \quad \kappa \times C_{old} = 2.1 \times 4.425\text{pF} = 9.2925\text{pF}$$

The breakdown field, and hence the maximum voltage, increase by a factor of 25, so the new maximum voltage is $V_{new} = 25 \times 12\text{kV} = 300\text{kV}$. The new maximum charge, $Q_{new} = C_{new}V_{new}$, is a factor of 2.1×25 greater than the old charge,

$$Q_{new} \quad = \quad 2.1 \times 25 \times Q_{old} = 2.8\mu\text{C}$$

(c) *Before the insertion of the Teflon plate, the plates are set to a voltage of 24V, and then the battery is disconnected. What are the energies in the capacitor BEFORE and AFTER the Teflon is inserted?*

There are three equivalent formulas for the potential energy stored in a capacitor: $U = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{1}{C}Q^2$. The voltage V between the plates will change when the Teflon is inserted, but the charge Q remains constant; hence the most useful formula is $U = \frac{1}{2}\frac{1}{C}Q^2$.

The charge on the capacitor is $Q = CV = 4.425\text{pF} \times 24\text{V} = 106.2\text{pC}$. Calculating the numbers,

$$\begin{aligned} U_{before} &= \frac{1}{2}\frac{1}{4.425 \times 10^{-12}}(106.2 \times 10^{-12})^2 = 1.27\text{nJ} \\ U_{after} &= \frac{1}{2}\frac{1}{9.2925 \times 10^{-12}}(106.2 \times 10^{-12})^2 = 0.61\text{nJ} \end{aligned}$$

The potential energy *decreases* when the dielectric is inserted; the energy goes towards pulling the dielectric into the capacitor (see the challenge problem on worksheet E6).

(d) *Suppose that the Teflon sheet in part (c) were only 2mm thick, and placed halfway between the plates. Find the electric field everywhere in the capacitor, and the new capacitance.*

The arrangement is shown in figure 1. Before the dielectric is inserted, the electric field has magnitude $|\vec{E}| = \frac{24\text{V}}{4\text{mm}} = 6000$V/m. After the dielectric is inserted, the electric field in regions A and C (outside the dielectric) remains unchanged, $E_A = 6000$V/m. In region B (inside the dielectric), the electric field decreases by a factor of κ: $E_B = \frac{6000}{2.1}$V/m$= 2900$V/m.

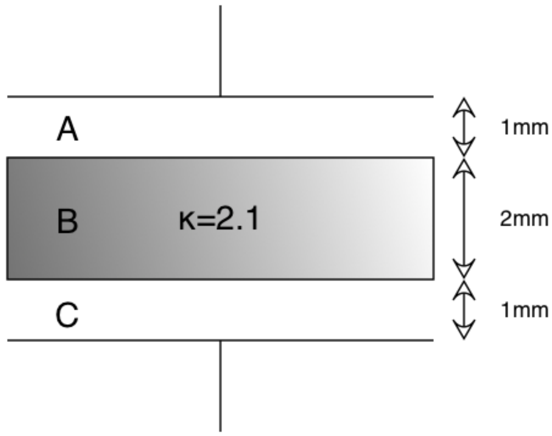

Figure 1: Capacitor with dielectric halfway between plates

Capacitance: Two methods to find the capacitance are 1. integrate $\Delta V = -\int \vec{E} \cdot d\vec{l}$, and 2. treat the arrangement as three capacitors in series, i.e. regions A, B, and C. I'll do the second method. Treating regions A and C like capacitors with $A = 20\text{cm}^2$, $d = 1$mm, and no dielectric, the capacitances of A and C are

$$C_A = C_C = \frac{\epsilon_0 A}{1\text{mm}}.$$

Treating region B like a capacitor with $A = 20\text{cm}^2$, $d = 2$mm, and dielectric constant $\kappa = 2.1$,

$$C_B = \frac{\kappa \epsilon_0 A}{2\text{mm}}$$

Now add these three values following the prescription for capacitors in series:

$$\begin{aligned}
\frac{1}{C} &= \frac{1}{C_A} + \frac{1}{C_B} + \frac{1}{C_C} \\
&= \frac{1\text{mm}}{\epsilon_0 A} + \frac{2\text{mm}}{\kappa \epsilon_0 A} + \frac{1\text{mm}}{\epsilon_0 A} \\
&= \frac{1}{\epsilon_0 A}\left(2\text{mm} + \frac{2\text{mm}}{\kappa}\right) \\
\Rightarrow C &= \frac{8.85 \times 10^{-12} \times 0.002}{(2 + \frac{2}{\kappa}) \times 10^{-3}} = 6\text{pF}
\end{aligned}$$

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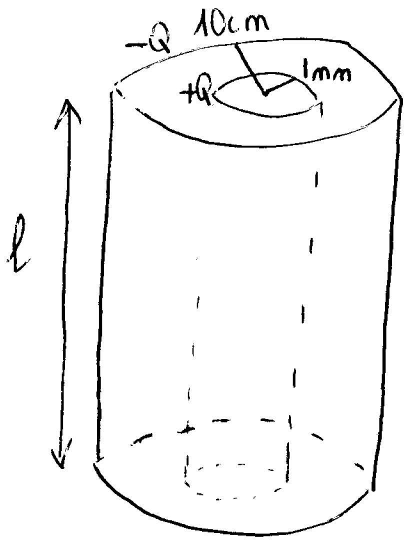
Problem # 2



* This field between $r = 1mm$ and $r = 10cm$ is $E = \frac{Q}{2\pi r h \varepsilon_0}$ using Gauss's Law with a surface = cylinder of radius r, length h.

* To find the capacitance $C = \frac{Q}{V}$, we need the difference of potential V

$$\Rightarrow V = \int_+^- \vec{E} \cdot d\vec{l} = \int_{0.001}^{0.1} \frac{Q}{2\pi r h \varepsilon_0} dr = \frac{Q}{2\pi h \varepsilon_0} \ln\left(\frac{0.1}{0.001}\right)$$

$$V = \frac{Q}{2\pi h \varepsilon_0} \ln(100)$$

So $C = \frac{2\pi h \varepsilon_0}{\ln(100)}$ so $\boxed{\frac{C}{h} = \frac{2\pi\varepsilon_0}{\ln(100)}}$ $\boxed{\frac{C}{h} = 1.2 \cdot 10^{-11} F.m^{-1}}$

* To find V such as $E = 3.10^6$ N/C at $r = 5mm$, we can write E in terms of V and r:

$$E = \frac{Q}{2\pi r h \varepsilon_0} = \frac{2\pi h \varepsilon_0 V / \ln(100)}{2\pi r h \varepsilon_0} = \frac{V}{r \ln(100)}$$

So $\underbrace{E}_{3.10^6 N/C} = \frac{V}{\underbrace{r}_{5.10^{-3} m} \ln(100)} \Rightarrow V = E r \ln(100)$

$$\boxed{V = 69.10^3 V}$$

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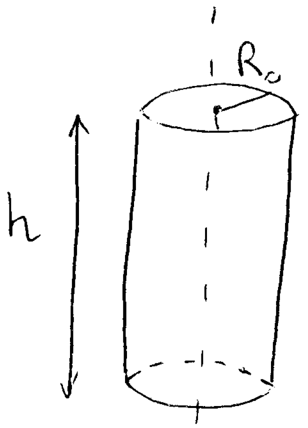
Problem # 3



charge density : $\rho_E(R) = \rho_0 \left(\frac{R}{R_0}\right)^2$.

* Field $\vec{E}$ inside the cylinder

Use Gauss's Law : surface = cylinder of radius $R < R_0$

So $\phi(\vec{E}) = \frac{Q_{enc}}{\varepsilon_0}$ with $Q_{enc} = \int \rho_E(R)\, dV = \int_0^R \rho_E(R)\, 2\pi R h\, dR$

$$\begin{cases} Q_{enc} = \dfrac{2\pi h \rho_0 R^4}{4 R_0^2} \\ \phi(\vec{E}) = E\, 2\pi R h \end{cases} \Longrightarrow \boxed{E = \frac{\rho_0 R^3}{4 \varepsilon_0 R_0^2}}$$ within the cylinder.

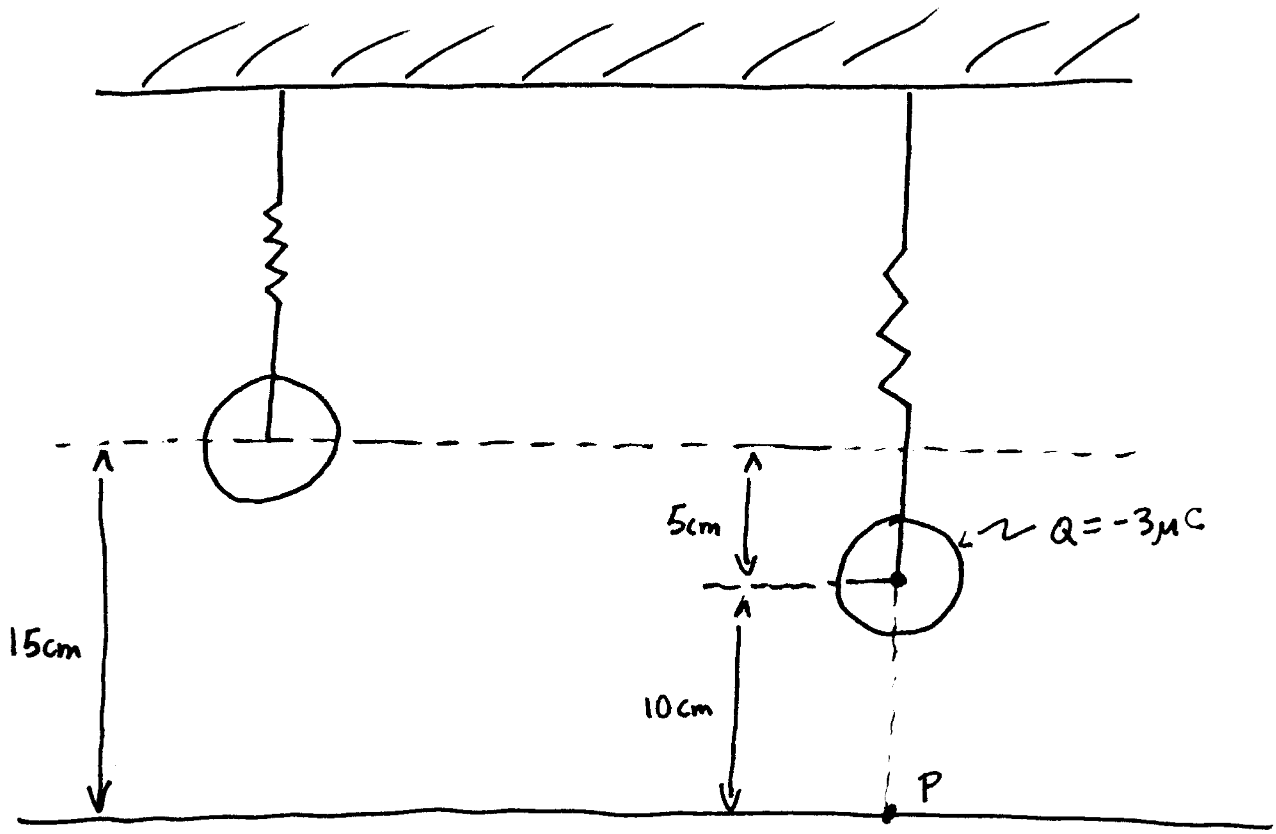
* Field $\vec{E}$ outside : use Gauss's Law again

This time, take surface = cylinder of radius $R > R_0$

so $\phi(\vec{E}) = \frac{Q_{enc}}{\varepsilon_0}$ with $Q_{enc} = \int_0^{R_0} \rho_E(R)\, 2\pi R h\, dR$

$$\begin{cases} Q_{enc} = \dfrac{2\pi h \rho_0 R_0^2}{4} \\ \phi(\vec{E}) = E\ 2\pi R h \end{cases} \Longrightarrow \boxed{E = \frac{\rho_0 R_0^2}{4 \varepsilon_0 R}}$$ outside of the cylinder.

Problem #4 Solution



1) Since there is no energy loss, we can conclude that the system will keep oscillating with a constant amplitude 5.0cm.

2) Using harmonic oscillator techniques, we get that the distance parametrized by the sphere is given by

$$y = (0.050\,m)\cos(\omega t) \longrightarrow ①$$

3) Note that the angular frequency of the sphere is

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{126\ N/m}{0.65\ kg}} = 13.9\ \frac{rad}{s} \longrightarrow ②$$

4) with ② in ① we get

$$y = (0.050\,m)\cos(13.9\,t) \longrightarrow ③$$

5) Now, we can get the distance from the point right below the sphere (P) and the sphere as follows (refer to figure)

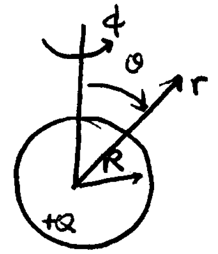
$$r = 0.150\,m - (0.050\,m)\cos(13.9t) \longrightarrow (4)$$

6) To measure the electric field at P, let us use techniques from electrostatics

$$E = K\frac{Q}{r^2} = \frac{(9.0\times10^{9}\ \frac{N\cdot m^2}{C^2})(3.0\times10^{-6}\,C)}{(0.150\,m - (0.050\,m)\cos(13.9t))^2}$$

$$= \frac{2.70\times10^{4}}{(0.150 - 0.050\cos(13.9t))^2}\ \frac{N}{C}$$

and since we have chosen to use a test charge that is positive, the electric field points upward at P prob. 5.



(a) 6pt. energy density $u_E(x,y,z) = \frac{\epsilon_0}{2}|\mathbb{E}(x,y,z)|^2$ 1pt

$$\mathbb{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & \text{, for } r>R \\ 0 & \text{, for } r<R \end{cases}$$

$$\therefore |\mathbb{E}|^2 = \begin{cases} \left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^4} & , r>R \\ 0 & , r<R. \end{cases} \Rightarrow u_E(r,\theta,\phi) = \frac{\epsilon_0}{2}\cdot\frac{Q^2}{16\pi^2\epsilon_0^2}\cdot\frac{1}{r^4}$$

$$= \begin{cases} \frac{Q^2}{32\pi^2\epsilon_0}\frac{1}{r^4} & r>R \quad 2pt \\ 0 & , r<R \quad 1pt. \end{cases}$$

$\therefore$ total potential energy $= \int_{\text{all space}} u_E \cdot dV = \int_R^\infty \left(\frac{Q^2}{32\pi^2\epsilon_0}\frac{1}{r^4}\right)\cdot 4\pi r^2 dr$

$$= \frac{Q^2}{8\pi\epsilon_0}\int_R^\infty \frac{1}{r^2}dr = \frac{-Q^2}{8\pi\epsilon_0}\frac{1}{r}\Big|_R^\infty = \underline{\frac{Q^2}{8\pi\epsilon_0 R}} \quad 2pt$$

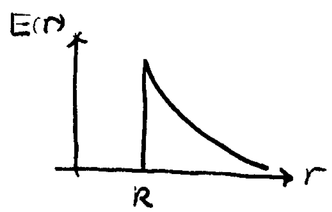
(b) 3pt work needed to bring charge dQ to the surface of the sphere w/ charge Q is $dW = V(Q)\,dQ = \frac{Q}{4\pi\epsilon_0 R}dQ$. 1 pt

$\rightarrow$ total work done $= \int_0^Q dW = \int_0^Q \frac{Q}{4\pi\epsilon_0 R}dQ = \frac{1}{4\pi\epsilon_0 R}\frac{Q^2}{2}\Big|_0^Q = \underline{\frac{Q^2}{8\pi\epsilon_0 R}}$ 2pt.

which is the same as the total potential energy. 1pt

(c) 7pt By Gauss's law.

$\mathbb{E}(r>R) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$. 2pt, $\mathbb{E}(r<R) = 0$. 1 pt
($\mathbb{E}=0$ inside conductor)



$r>R$

$$V(r) = \int_\infty^r -\mathbb{E}(r')\cdot d\mathbf{r}' = \int_\infty^r -\frac{Q}{4\pi\epsilon_0 r'^2}\underbrace{\hat{r}'\cdot d\mathbf{r}'}_{=dr'} = \int_\infty^r \frac{-Q}{4\pi\epsilon_0 r'^2}dr' = \frac{Q}{4\pi\epsilon_0 r}$$

while V = const inside conductor

$\Rightarrow$ for $r<R$. $V(r) = V(R) = \frac{Q}{4\pi\epsilon_0 R}$.

$$V(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r} & r>R \quad 2pt \\ \frac{Q}{4\pi\epsilon_0 R}. & r<R. \quad 2pt \end{cases}$$

(d) 4pt at the surface $V(R) = \frac{Q}{4\pi\epsilon_0 R} \Rightarrow Q = 4\pi\epsilon_0 R\,V(R) = 2.42\times10^{-8}C$

$\therefore$ the surface charge density $\sigma = \frac{Q}{4\pi R^2} = \frac{\epsilon_0 V(R)}{R} = \frac{\epsilon_0\cdot 680V}{32\times10^{-2}m}$

$= 1.88\times10^{-8}\,C/m^2$ 2pt

$$V(r) = \frac{Q}{4\pi\epsilon_0 r}$$

$$\Rightarrow r = \frac{Q}{4\pi\epsilon_0 V(r)} = \frac{2.42 \times 10^{-8} C}{4 \cdot \pi \cdot \epsilon_0 \cdot 25V} = 8.70 \text{ m}$$ 2pt