EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 1
Department of Electrical Engineering and Computer Sciences 21 September 2010
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Rack FIRST Name D

Lab Time ?

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.

- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.

- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided 8.5" $\times$ 11" sheet of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.

- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.

- Please write neatly and legibly, because *if we can't read it, we can't grade it.*

- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*

- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.

- We hope you do a *fantastic* job on this exam.

MT1.1 (45 Points) The instantaneous position of a particle on the complex plane is described as follows:

$$\forall t \in \mathbb{R}, \quad z(t) = \sin(t)\, e^{i\,t}.$$

(a) (10 Points) Determine reasonably simple expressions for the instantaneous Cartesian (rectangular) coordinates of the particle; that is determine $x(t)$ and $y(t)$, where $z(t) = x(t) + i\,y(t)$.

Method 1:

$$z(t) = \sin(t)(\cos t + i\sin t) = \sin t\cos t + i\sin^2 t = \overbrace{\boxed{\frac{\sin(2t)}{2}}}^{x(t)} + i\overbrace{\boxed{\frac{1-\cos(2t)}{2}}}^{y(t)}$$

Method 2:

$$z(t) = \frac{e^{it}-e^{-it}}{2i}\,e^{it} = \frac{e^{i2t}-1}{2i} = \frac{\cos(2t)+i\sin(2t)-1}{2i}.$$

Multiply by $\frac{-i}{-i}$ to remove the i from the denominator:

$$z(t) = \frac{-i\cos(2t)+(-i)\,i\sin(2t)-(-i)}{2i(-i)} = \frac{\sin(2t)}{2} + i\,\frac{1-\cos(2t)}{2},$$

which is the same as above.

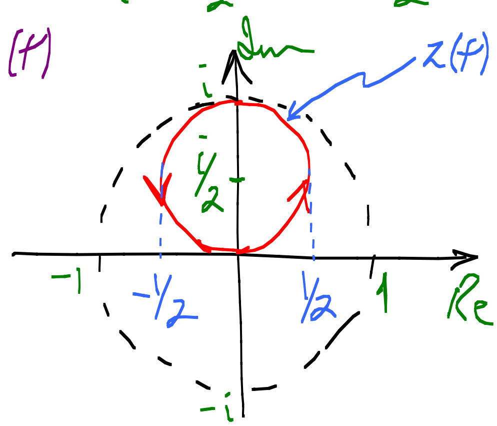
(b) (15 Points) Provide a well-labeled plot of the *trajectory* of the particle on the complex plane. That is, indicate the path and the direction of the particle's motion. To receive credit, you must provide a succinct, but clear and convincing explanation of your work. Please note that we are *not* asking you to plot $x(t)$ and $y(t)$.

From part (a) we know that $z(t) = \frac{e^{i2t}-1}{2i} = \frac{-ie^{i2t}+i}{2}$ $\xrightarrow{\text{We know that } -i = e^{-i\pi/2}}$

$$z(t) = \underbrace{\boxed{\frac{e^{i(2t-\pi/2)}}{2}}}_{\varphi(t)} + \frac{i}{2}$$

If $\varphi(t) \triangleq \frac{1}{2}e^{i(2t-\pi/2)}$ denotes the instantaneous position of a particle on the complex plane, it means that the particle traverses a circle of radius $1/2$ at the rate of 2 radians/sec [1] counter-clockwise, starting from $\varphi(0) = \frac{1}{2}e^{-i\pi/2} = \frac{-i}{2}$.

The $z(t) = \varphi(t) + \frac{i}{2}$, so the trajectory for $z(t)$ is simply the one for $\varphi(t)$ shifted up by $i/2$, as shown by the arrowed circle here (in red):



[1] See part (c)

(c) (10 Points) Show that the particle's position exhibits periodic behavior, and determine its fundamental period p and fundamental frequency ω_0.

We showed that $z(t) = \frac{1}{2i} e^{i2t} - \frac{1}{2i}$. It should be clear that z is periodic because e^{i2t} is a phasor rotating counterclockwise in a circle twice as fast as the phasor e^{it}. So, we expect the fundamental frequency to be 2 radians per second, corresponding to a period of $p = \frac{2\pi}{2} = \pi$ seconds.

Let's s aw this more formally: We want to find a $p > 0$ such that $z(t+p) = z(t)$

$$z(t+p) = \cancel{\frac{1}{2i}} e^{i2(t+p)} \cancel{- \frac{1}{2i}} = \cancel{\frac{1}{2i}} e^{i2t} \cancel{- \frac{1}{2i}} \Rightarrow \cancel{e^{i2t}} e^{i2p} = \cancel{e^{i2t}} \Rightarrow e^{i2p} = 1 = e^{i2\pi k} \Rightarrow$$

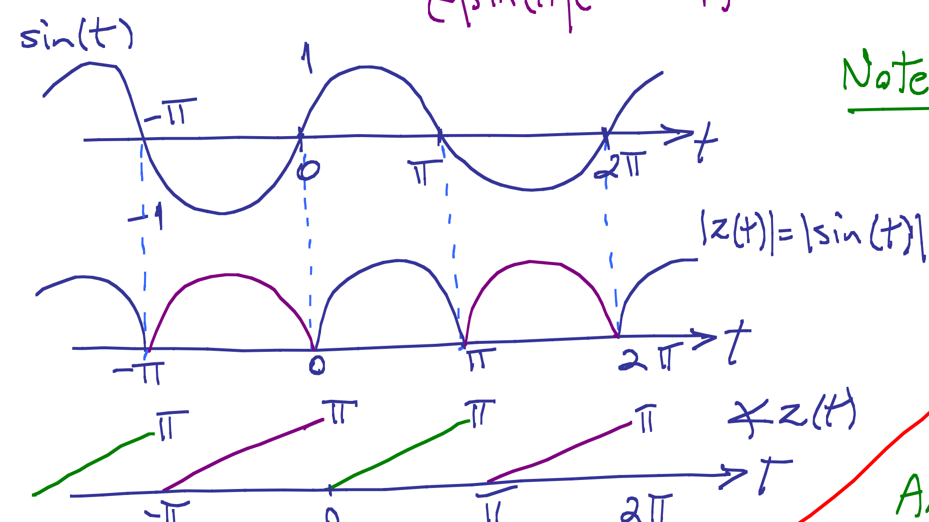
$$\cancel{2}p = \cancel{2}\pi k \Rightarrow p = \pi k.$$ The smallest positive k is 1 $\Rightarrow$ $p = \pi$

(d) (10 Points) Provide well-labeled plots of $|z(t)|$ and $\angle z(t)$.

$z(t) = \sin(t) e^{it} \Rightarrow |z(t)| = |\sin(t)|$

As for the phase $\measuredangle z(t)$, we notice that

$$z(t) = \begin{cases} |\sin(t)| e^{it} & \text{if } \sin(t) > 0 \\ -|\sin(t)| e^{it} & \text{if } \sin(t) < 0 \end{cases} = \begin{cases} |\sin(t)| e^{it} & \sin(t) > 0 \\ |\sin(t)| e^{i(t+\pi)} & \sin(t) < 0 \end{cases}$$



Note: In the interval $-\pi < t < \pi$
$\sin t > 0$ for $0 < t < \pi$ and
$\sin t < 0$ for $-\pi < t < 0$
so

$$\measuredangle z(t) = \begin{cases} t & 0 < t < \pi \\ t + \pi & -\pi < t < 0 \end{cases}$$

and this periodically repeats w/ period 2π.

Another way is to plot $|z(t)|$ & $\measuredangle z(t)$ for $0 < t < \pi$ and use the π-periodicity of z (which we established in part (c)) to draw the rest.

The $|z(t)|$ & $\measuredangle z(t)$ plots obtained this way confirm the π-periodicity of $z(t)$.

MT1.2 (30 Points) Consider two complex numbers x and y.

(a) (7 Points) Prove that $(x + y)^* = x^* + y^*$; that is, the complex conjugate of the sum of two numbers is the sum of their individual complex conjugates.

Let $x = x_R + i x_I$ and $y = y_R + i y_I \implies x+y = (x_R + y_R) + i(x_I + y_I) \implies$

$$(x+y)^* = (x_R + y_R) - i(x_I + y_I) = \underbrace{(x_R - i x_I)}_{x^*} + \underbrace{(y_R - i y_I)}_{y^*} \implies$$

$$(x+y)^* = x^* + y^*$$

We can easily extend this result to a sum of more than two complex numbers. For example, using the same method as the one above we can show that $\left(\sum_k z_k\right)^* = \sum_k z_k^*$

(b) (7 Points) Prove that $(xy)^* = x^* y^*$; that is, the complex conjugate of the product of two numbers is the product of their individual complex conjugates.

For this part we first express x and y in polar form; as a rule of thumb, multiplying two complex numbers is more easily done if they're expressed in polar form. In particular, let $x = R_x e^{i\theta_x}$, where $R_x = |x|$ and $\theta_x = \measuredangle x$, and let $y = R_y e^{i\theta_y}$, where $R_y = |y|$ and $\theta_y = \measuredangle y$. Then $xy = R_x e^{i\theta_x} R_y e^{i\theta_y} = R_x R_y e^{i(\theta_x + \theta_y)}$

Accordingly, $(xy)^* = R_x R_y e^{-i(\theta_x+\theta_y)} = \underbrace{\left(R_x e^{-i\theta_x}\right)}_{x^*}\underbrace{\left(R_y e^{-i\theta_y}\right)}_{y^*}$. So we've shown that $(xy)^* = x^* y^*$

We can extend this to a product of more than two numbers easily. The result is $\left(\prod_k z_k\right)^* = \prod_k z_k^*$

(c) (16 Points) Consider a cubic polynomial

$$Q(z) = \sum_{k=0}^{3} a_k\, z^k = a_0 + a_1\, z + a_2\, z^2 + a_3\, z^3,$$

where every coefficient a_k is real.

(i) (8 Points) Prove that if z is a complex root of the polynomial (i.e., $Q(z) = 0$, and $z \notin \mathbb{R}$), then so is z^*.

z is a root of $Q(z) \Rightarrow Q(z) = \sum_{k=0}^{3} a_k z^k = 0 \Rightarrow$
$Q^*(z) = \left(\sum_{k=0}^{3} a_k z^k\right)^* = 0$. Using the result of part (a), we know that $Q^*(z) = \sum_{k=0}^{3} (a_k z^k)^* = 0$. Now apply the result of part (b): $Q^*(z) = \sum_{k=0}^{3} \underbrace{a_k^* (z^k)^* = \sum_{k=0}^{3} a_k}_{a_k^* = a_k \text{ b/c } a_k \in \mathbb{R}} (z^*)^k = Q(z^*) = 0$

We have shown that
if $Q(z) = \sum_{k=0}^{3} a_k z^k = 0$, $\forall a_k \in \mathbb{R} \Rightarrow Q(z^*) = 0$

We can extend this result to any polynomial that has real coefficients. If $\sum_{k=0}^{N} a_k z^k = 0,\ a_k \in \mathbb{R} \Rightarrow \sum_{k=0}^{N} a_k (z^*)^k = 0$

Interpretation: The roots of any polynomial having real coefficients are either real, or they appear as complex conjugate pairs.

(ii) (8 Points) Prove that $Q(z)$ has at least one real root.

$Q(z)$ has three roots, so we can write it as

$Q(z) = a_3 (z - z_1)(z - z_2)(z - z_3)$, where $Q(z_1) = Q(z_2) = Q(z_3) = 0$.

If $z_1 \in \mathbb{C} - \mathbb{R}$ is a root of $Q(z)$, then $z_1^* \in \mathbb{C} - \mathbb{R}$ is also a root. Call this root z_2; that is $z_2 = z_1^* \Rightarrow Q(z) = a_3 (z - z_1)(z - z_1^*)(z - z_3)$.

Per force, $z_3 \in \mathbb{R}$ because if $z_3 \in \mathbb{C} - \mathbb{R}$, then z_3^* must also be a root; but the other two roots are taken up by z_1 and z_1^*, a contradiction.

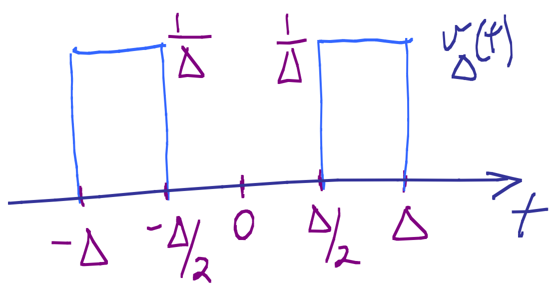
Therefore $z_3 \in \mathbb{R}$.

Generally, any odd-order polynomial having real coefficients has at least one real-valued root.

MT1.3 (30 Points) For $\Delta > 0$, let $v_\Delta(t)$ be defined by

$$v_\Delta(t) = \begin{cases} 0 & \text{if } t < -\Delta \\ \frac{1}{\Delta} & \text{if } -\Delta \le t \le -\frac{\Delta}{2} \\ 0 & \text{if } -\frac{\Delta}{2} < t < \frac{\Delta}{2} \\ \frac{1}{\Delta} & \text{if } \frac{\Delta}{2} \le t \le \Delta \\ 0 & \text{if } \Delta < t. \end{cases}$$

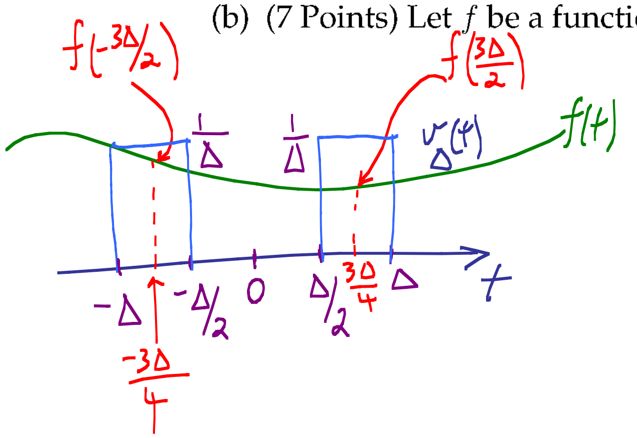
(a) (7 Points) Provide a well-labeled plot of v_Δ.



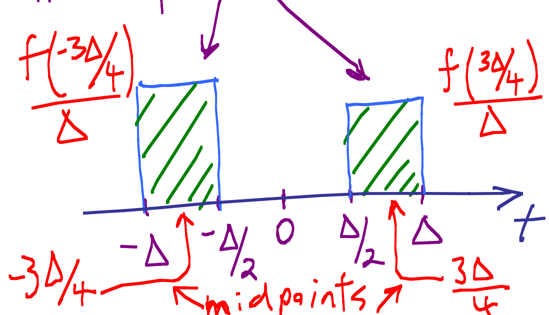
(b) (7 Points) Let f be a function that is continuous at 0. Determine

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} f(t)\, v_\Delta(t) dt\,.$$

For Δ small enough, $\int_{-\infty}^{\infty} f(t) v_\Delta(t) dt$ is approximately equal to the area shown in the figure below:



For Δ small enough, the height of each box is approximately $\frac{1}{\Delta}$ scaled by the value of the function f at the midpoint of the region of support of the respective box.



$$\int_{-\infty}^{\infty} f(t) v_\Delta(t) dt \approx \frac{\Delta}{2}\left[\frac{f(-3\Delta/4)}{\Delta} + \frac{f(3\Delta/4)}{\Delta}\right] = \frac{1}{2}\left[f\left(\frac{-3\Delta}{4}\right) + f\left(\frac{3\Delta}{4}\right)\right]$$

But f is continuous at $0 \Rightarrow \lim_{\Delta \to 0}\left[f(-3\Delta/4) + f(3\Delta/4)\right] = f(0) + f(0) = 2f(0)$

$$\Rightarrow \lim_{\Delta \to 0} \int_{-\infty}^{\infty} f(t) v_\Delta(t) dt = f(0)$$

(c) (6 Points) From the preceding part, how should we represent

$$\lim_{\Delta \to 0} v_\Delta(t)\ ,$$

from the point of view of how it behaves inside integrals?

We've seen that $\lim_{\Delta\to 0}\int_{-\infty}^{\infty} f(t)\, v_\Delta(t)\, dt = \int_{-\infty}^{\infty} f(t)\left[\lim_{\Delta\to 0} v_\Delta(t)\right] dt = f(0)$

But we also know that $\int_{-\infty}^{\infty} f(t)\,\delta(t)\, dt = f(0)$

$\Rightarrow \boxed{\lim_{\Delta\to 0} v_\Delta(t) = \delta(t)}$

The equality should not be understood to be pointwise; rather it is meant in the following sense: $\int_{-\infty}^{\infty} f(t)\left[\lim_{\Delta\to 0} v_\Delta(t)\right] dt = \int_{-\infty}^{\infty} f(t)\,\delta(t)\, dt$.

That is, $\lim_{\Delta\to 0} v_\Delta(t)$ and $\delta(t)$ behave the same way when they come in contact with a test function f inside an integral.

(d) (10 Points) Let f and g be two functions, each of which is continuous at $t = 0$ and at $t = 1$. Determine

Let $h(t) = f(t)\, g(t)$

$$\int_{-\infty}^{+\infty} [\delta(2t) + \delta(t-1)]\, \underbrace{f(t)g(t)}\, dt,$$

where δ denotes the Dirac delta function.

$$\int_{-\infty}^{\infty} [\delta(2t) + \delta(t-1)]\, f(t) g(t)\, dt = \int_{-\infty}^{\infty} [\delta(2t) + \delta(t-1)]\, h(t)\, dt.$$ But $\delta(2t) = \frac{1}{2}\delta(t)$, so

$$\int_{-\infty}^{\infty} [\delta(2t) + \delta(t-1)]\, h(t)\, dt = \int_{-\infty}^{\infty} \left[\frac{1}{2}\delta(t) + \delta(t-1)\right] h(t)\, dt = \frac{h(0)}{2} + h(1) \Rightarrow$$

$$\boxed{\int_{-\infty}^{\infty} [\delta(2t) + \delta(t-1)]\, f(t) g(t) = \frac{f(0)g(0)}{2} + f(1)\, g(1)}$$

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Lab Time ?

Problem	Points	Your Score
Name	10	10
1	45	45
2	30	30
3	30	30
Total	**115**	115

You may or may not find the following information useful:

$$\cos(\alpha+\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta.$$

$$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta.$$

$$\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right).$$

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right).$$

$$\delta(at) = \frac{1}{|a|}\delta(t).$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1-2\sin^2\alpha \Rightarrow \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$