UCB Math 110, Fall 2010: Midterm 2

Prof. Persson, November 8, 2010

Name: Solutions

SID:

Section: Circle your discussion section below:

Sec	Time	Room	GSI
01	Wed 8am - 9am	87 Evans	D. Penneys
02	Wed 9am - 10am	2032 Valley LSB	C. Mitchell
03	Wed 10am - 11am	B51 Hildebrand	D. Beraldo
04	Wed 11am - 12pm	B51 Hildebrand	D. Beraldo
05	Wed 12pm - 1pm	75 Evans	C. Mitchell
07	Wed 2pm - 3pm	87 Evans	C. Mitchell
08	Wed 9am - 10am	3113 Etcheverry	I. Ventura
09	Wed 2pm - 3pm	3 Evans	D. Penneys
10	Wed 12pm - 1pm	310 Hearst	I. Ventura

Other/none, explain: ______________________

Grading

1	/ 18
2	/ 6
3	/ 6
4	/ 10
	/ 40

Instructions:

- One double-sided sheet of notes, no books, no calculators.
- Exam time 50 minutes, do all of the problems.
- You must justify your answers for full credit.
- Write your answers in the space below each problem.
- If you need more space, use reverse side or scratch pages. Indicate clearly where to find your answers.

1. (6 problems, 3 points each) Label the following statements as TRUE or FALSE, giving a short explanation (e.g. a proof or a counterexample).

a) Let $A, B \in \mathsf{M}_{5\times5}(R)$ such that $AB = -BA$. Then either A or B is non-invertible.

TRUE FALSE (circle one)

Taking determinants, we have

$\det(A)\det(B) = \det(AB) = \det(-BA)$
$= (-1)^5 \det(BA) = -\det(B)\det(A)$
$\Rightarrow 2\det(A)\det(B) = 0$
$\Rightarrow$ either $\det(A)$ or $\det(B) = 0$
$\Rightarrow$ either A or B not invertible.

b) Every matrix $A \in \mathsf{M}_{5\times5}(R)$ has an eigenvector in R^5.

TRUE FALSE (circle one)

Since the characteristic polynomial of A has degree 5, it must have at least one real root by the intermediate value theorem. Hence
$\exists \lambda \in \mathbb{R}$ s.t. $\ker(A-\lambda I) = N(A-\lambda I) \neq (0)$
$\Rightarrow \exists v \in \mathbb{R}^5$, $v \neq 0$, s.t. $Av = \lambda v$.
(A has an eigenvector in $\mathbb{R}^5$)

c) Let $A, B \in \mathsf{M}_{n\times n}(F)$, and suppose A is similar to B. Then A^k is similar to B^k for any positive integer k.

TRUE FALSE (circle one)

we induct on k.

<u>$k=1$</u>: $A \sim B \Rightarrow \exists Q \in M_n(F)$ invertible s.t.
$A = Q^{-1}BQ$.

<u>$k-1 \Rightarrow k$</u>: Suppose $A^{k-1} = Q^{-1}B^{k-1}Q$. Then
$A^k = Q^{-1}BQQ^{-1}B^{k-1}Q = Q^{-1}B^kQ$

$\Rightarrow A^k \sim B^k \ \forall k > 0$.

1. (cont'd)

d) If 0 is the only eigenvalue of a linear operator T, then T = 0.

TRUE $\boxed{\text{FALSE}}$ (circle one)

Consider $L_A: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Since A is upper triangular we see the eigenvalues are both 0,

e) If a matrix $A \in \mathsf{M}_{n \times n}(F)$ can be transformed into a diagonal matrix by a sequence of elementary row operations of type 3, then A is diagonalizable.

TRUE $\boxed{\text{FALSE}}$ (circle one)

The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable but can be reduced to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

f) Let V be a finite dimensional vector space and γ be a basis for V*. Then there exists a basis β for V such that $\beta^* = \gamma$.

$\boxed{\text{TRUE}}$ FALSE (circle one)

Consider γ^* a basis for V^{**}. We know the map $\tau: V \to V^{**}$ $\tau(x) = \hat{x}$ is an isomorphism ~~From~~ If $\gamma^{**} = \{\hat{x}_1, \ldots, \hat{x}_n\}$ we let

$\beta = \{x_1, \ldots, x_n\}$ where $x_i = \tau^{-1}(\hat{x}_i)$ b/c τ is an isomorphism β is a basis and it is easy to see $\beta^* = \gamma$.

2. (6 points) Find bases for the null space $\mathsf{N}(\mathsf{L}_A)$ and for the range $\mathsf{R}(\mathsf{L}_A)$ where

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 4 & -4 & 5 & -2 \\ 2 & -2 & -1 & -8 \end{pmatrix}. \to \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -3 & -6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis for $N(L_A)$: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$

Basis for $R(L_A)$: $\left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} \right\}$

3. (6 points) Let $\mathsf{V} = \mathsf{R}^2$ and define $\mathsf{f}, \mathsf{g} \in \mathsf{V}^*$ as follows:

$$\mathsf{f}(x, y) = x + y, \quad \mathsf{g}(x, y) = x - 2y.$$

Find a basis β for V such that its dual basis $\beta^* = (\mathsf{f}, \mathsf{g})$.

$\beta = \left\{ \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix} \right\}$

$x_1 + y_1 = 1$ | $x_2 + y_2 = 0$

$2x_1 \cancel{+} 2y_1 = 0$ | $x_2 - 2y_2 = 1$

Solve the system to get the desired result.

4. (10 points) Consider the linear operator T on $P_3(R)$ defined by

$$T(p(x)) = (x^2+1)p''(x).$$

Determine if T is diagonalizable, and if so, find a basis β for $P_3(R)$ such that $[T]_\beta$ is a diagonal matrix.

One can guess a basis of eigenvectors:

$\beta = \{1, x, 1+x^2, x+x^3\}$

β is a linearly independent set (the four polynomials have different degrees) and $|\beta| = 4$, so β is a basis for $P_3(\mathbb{R})$.

Now, observe that

$T(1) = 0$

$T(x) = 0$

$T(1+x^2) = 2\cdot(x^2+1) = 2(1+x^2)$

$T(x+x^3) = (x^2+1)(6x) = 6(x+x^3)$

So β is a basis of eigenvectors.

$$[T]_\beta = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 2 & \\ & & & 6 \end{pmatrix}.$$