UCB Math 110, Fall 2010: Midterm 1

Prof. Persson, October 4, 2010

Name: Solutions

SID: ____________________

Section: Circle your discussion section below:

Sec	Time	Room	GSI
01	Wed 8am - 9am	87 Evans	D. Penneys
02	Wed 9am - 10am	2032 Valley LSB	C. Mitchell
03	Wed 10am - 11am	B51 Hildebrand	D. Beraldo
04	Wed 11am - 12pm	B51 Hildebrand	D. Beraldo
05	Wed 12pm - 1pm	75 Evans	C. Mitchell
07	Wed 2pm - 3pm	87 Evans	C. Mitchell
08	Wed 9am - 10am	3113 Etcheverry	I. Ventura
09	Wed 2pm - 3pm	3 Evans	D. Penneys
10	Wed 12pm - 1pm	310 Hearst	I. Ventura

Other/none, explain: ____________________

Grading

1	/ 18
2	/ 12
3	/ 10
	/ 40

Instructions:

- One double-sided sheet of notes, no books, no calculators.
- Exam time 50 minutes, do all of the problems.
- You must justify your answers for full credit.
- Write your answers in the space below each problem.
- If you need more space, use reverse side or scratch pages. Indicate clearly where to find your answers.

1. (6 problems, 3 points each) Label the following statements as TRUE or FALSE, giving a short explanation (e.g. a proof or a counterexample).

a) Let $\mathsf{V} = R^n$ and $\mathsf{W} = R$. Then $\mathcal{L}(\mathsf{V}, \mathsf{W})$ is isomorphic to $\mathsf{P}_n(R)$.

F

$\dim(\mathcal{L}(V,W)) = \dim(V)\dim(W) = n$

$\dim(P_n(\mathbb{R})) = n+1.$

since the dimensions are not the same, ~~they cannot~~ $\mathcal{L}(V,W)$ is not isom. to $P_n(\mathbb{R})$.

b) Let V be a vector space and $\mathsf{T}, \mathsf{U} : \mathsf{V} \to \mathsf{V}$ be two linear operators. Then $\mathsf{N}(\mathsf{U}) \subseteq \mathsf{N}(\mathsf{TU})$.

T. Let $v \in N(U)$. then $U(v) = 0$.

$\Rightarrow (TU)(v) = T(Uv) = T(0) = 0$

as T is linear.

$\Rightarrow v \in N(TU)$.

c) Let V be a vector space and $\mathsf{T}, \mathsf{U} : \mathsf{V} \to \mathsf{V}$ be two linear operators. Then $\mathsf{R}(\mathsf{U}) \subseteq \mathsf{R}(\mathsf{UT})$.

F. Suppose $T: \mathbb{R} \to \mathbb{R}$ is the zero lin. trans., i.e., $Tx = 0$ $\forall x \in \mathbb{R}$.

Suppose $U: \mathbb{R} \to \mathbb{R}$ is the identity lin. trans., i.e, $Ux = x$ $\forall x \in \mathbb{R}$.

Then $R(U) = \mathbb{R} \not\subseteq \{0\} = R(UT)$

<u>Note</u>: that $R(UT) \subseteq R(U)$. . . .

1. (cont'd)

True **d)** The set $S = \{p \in \mathsf{P}(F) : p'(0) = p(0)\}$ is a subspace of $\mathsf{P}(F)$.

Consider $T : P(F) \to F$ with $T(p(x)) = p'(0) - p(0)$

T is a Linear Transformation ~~so~~ and

$S = N(T)$. Thus S is a ~~linear~~ Subspace of $P(F)$

False **e)** Let $\mathsf{T} : R^2 \to R^2$ be a linear transformation. Then $R^2 = \mathsf{N}(\mathsf{T}) \oplus \mathsf{R}(\mathsf{T})$.

Consider L_A with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$R(L_A) = N(L_A) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

thus $R^2 \neq N(T) \oplus R(T)$.

True **f)** Let W_1 and W_2 be 3-dimensional subspaces of R^5. Then W_1 and W_2 must have a common nonzero vector.

Let β_1 and β_2 be bases for W_1 and W_2 respectively, with $\beta_1 = \{u_1, u_2, u_3\}$ $\beta_2 = \{v_1, v_2, v_3\}$. if $\beta_1 \cap \beta_2 \neq \phi$ we are done.

if $\beta_1 \cap \beta = \phi$ $\beta_1 \cup \beta_2$ is Linearly dependent.

thus $\exists$ $a_i\ b_i \in R$ Not all 0 st

$$a_1u_1 + a_2u_2 + a_3u_3 + b_1v_1 + b_2v_2 + b_3v_3 = 0$$

thus $a_1u_1 + a_2u_2 + a_3u_3 = \ldots$ giving the

2. (12 points) Let $\mathsf{T} : \mathsf{M}_{2\times 2}(R) \to \mathsf{M}_{2\times 2}(R)$ be defined by

$$\mathsf{T}(A) = BA - A^t \qquad \text{where } B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

a) Prove that T is a linear transformation.

$$T(aA+C) = B(aA+C) - (aA+C)^t$$

$$= B(aA) + BC - (aA)^t - C^t$$

$$= aBA - aA^t + BC - C^t$$

$$= aT(A) + T(C)$$

b) Find bases for $\mathsf{N}(\mathsf{T})$ and $\mathsf{R}(\mathsf{T})$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad T(A) = BA - A^t = \begin{pmatrix} a+2c & b+2d \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$= \begin{pmatrix} 2c & b+2d-c \\ c-b & 0 \end{pmatrix}$$

$$T(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow b = c = d = 0 \Rightarrow \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \text{ basis for } N(T)$$

$$R(T) = \operatorname{span}\left(\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \right\} \right)$$

$$= \operatorname{span}\Big(\underbrace{\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}}_{\text{basis for } R(T)} \Big)$$

3. (10 points) Let V be a vector space, and $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ a linear operator. Suppose $x \in \mathsf{V}$ is such that $\mathsf{T}^m(x) = 0$ but $\mathsf{T}^{m-1}(x) \neq 0$ for some positive integer m. Show that $\{x, \mathsf{T}(x), \mathsf{T}^2(x), \dots, \mathsf{T}^{m-1}(x)\}$ is linearly independent.

~~We prove by induction.~~

~~We assume $\{T^{m-k}x, \dots, T^{m-1}x\}$ is a LI set for k~~

Show: $T^k x \notin \operatorname{span}(T^{k+1}x, \dots, T^{m-1}x) \quad \forall k \geq 0$

If $T^k x = \alpha_1 T^{k+1} x + \dots + \alpha_{m-k-1} T^{m-1} x$

then if we aply T^{m-k-1} to both sides we see that

$T^{m-1} x = 0$ which contradicts our original essumption

$\therefore T^k x \notin \operatorname{Span}(T^{k+1}x, \dots, T^{m-1}x) \quad \forall k$

$\Rightarrow \{T^k x, \dots, T^{m-1}x\}$ is a LI set $\forall k \geq 0$

$\therefore \{x, Tx, \dots, T^{m-1}x\}$ is a LI set