EE 42/100 Mid-term 1

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1 Solution 1

There are several ways of solving this problem.The most obvious ways in ordered from least work to most work is listed below:

a. Replace black-box with Thevenin-equivalent circuit;apply voltage divider formula twice; solve the two system equation

b. Replace black-box with Thevenin-equivalent circuit;find currents for the two cases; note that it should be the same for the circuit; apply Ohm's law twice; solve resulting two system equation

c. Replace black-box with Norton-equivalent circuit; eliminate the common unknown variable...you get the idea.

I'm going to go ahead and list out the solution using approach a.

$$v_m = \frac{R}{(R_{th} + (R_1 + R))} * V_{th} \quad (1)$$

$$2 = \frac{1}{(R_{th} + (1 + 1)} * V_{th} \quad (2)$$

$$3 = \frac{2}{(R_{th} + (1 + 2))} * V_{th} \quad (3)$$

On simplifying the above equations, you get

$$2R_{th} + 4 = V_{th} \quad (4)$$

$$3R_{th} + 9 = 2V_{th} \quad (5)$$

On solving these equations, you should get

$$\boxed{R_{th} = 1\Omega; V_{th} = 6V; i_{Norton} = i_{sc} = 6A}$$

2 Solution 2

This problem is very similar to the one you had on your problem set; the sole exception being that now the ”‘rungs”’ of your ladder have three circuit elements as opposed to only two and its in the shape of the letter ’C’. Before completing the problem, it is imperative to note that adding one more rung to this infinite ladder is not going to affect the effective capacitance in any way between a and b.As a sanity check, once you make the substituion, your final circuit will look like a square in which the rung remains the same and the last edge is replaced by C_{eq}.Please be careful with regards to which part of the circuit you replace with the effective capacitance. You can replace the infinite ladder with C_{eq} to get C, C and C_{eq} to be in series. Let us call its effective capacitance to be C_{rung}

$$\frac{1}{C_{rung}} = \frac{1}{C} + \frac{1}{C} + \frac{1}{C_{eq}} \tag{6}$$

$$C_{rung} = \frac{CC_{eq}}{(2C_{eq} + C)} \tag{7}$$

Now, from the reduced circuit, note that C_{rung} is in parallel with the last C.Therefore,

$$C_{eq} = C_{rung} + C \tag{8}$$

$$C_{eq} = \frac{CC_{eq}}{(2C_{eq} + C)} + C \tag{9}$$

$$C_{eq} = \frac{CC_{eq} + 2CC_{eq} + C^2}{2C_{eq} + C} \tag{10}$$

$$2C_{eq}^2 + \cancel{CC_{eq}} = \cancel{CC_{eq}} + 2CC_{eq} + C^2 \tag{11}$$

$$2C_{eq}^2 - 2CC_{eq} - C^2 = 0 \tag{12}$$

$$C_{eq} = \frac{-(-2C) \pm \sqrt{(-2C)^2 - 4(2)(-C^2)}}{2(2)} \tag{13}$$

$$C_{eq} = \frac{2C \pm \sqrt{12C^2}}{4} \tag{14}$$

$$C_{eq} = \frac{1 \pm \sqrt{3}}{2}C \tag{15}$$

Notice that the capacitance is a physical and very real value which implies that it must be positive. Therefore,

$$\boxed{C_{eq} = \frac{1+\sqrt{3}}{2}C}$$

(Note that Farad is implied in the C)

3 Solution 3

1. From a quick inspection of the circuit, we have five nodes and three current loops. It looks like it would be easier to use mesh currents, since we would only have three unknowns, versus four for node voltages. If we call the top left mesh i_1, the bottom left i_2, and the bottom right i_3, then our equations are
$$i_1R + i_1R + (i_1 - i_2)R = 0 \tag{16}$$
$$(i_2 - i_1)R + (i_2 - i_3)R - 5mV = 0 \tag{17}$$
$$i_3R + 3v + (i_3 - i_2)R = 0 \tag{18}$$
Noting that $R = 1\Omega$ and $v = (i_2 - i_3)R$, we can simplify:
$$3i_1 - i_2 = 0 \tag{19}$$
$$-i_1 + 2i_2 - i_3 = 5m \tag{20}$$
$$i_3 + 3(i_2 - i_3) + i_3 - i_2 = 2i_2 - i_3 = 0 \tag{21}$$
The solutions are $i_1 = -5mA$, $i_2 = -15mA$, and $i_3 = -30mA$. Thus, by KVL we can see that
$$v_a = 5mV - i_1R = 10mV \tag{22}$$
Similarly, v_b is defined to be $3v$, where
$$v = (i_2 - i_3)R = 15mV \tag{23}$$
$$\boxed{v_m = v_a - v_b = 10mV - 45mV = -35mV}$$

2. We can quickly rule out configurations 1 and 2. The first one uses positive feedback, which does not allow us to control our desired gain. In addition, the first op amp's output is shorted to the inverting input of the second op amp, forcing it to ground to match that of the noninverting input. Thus, our v_{out} is not well-defined. The second configuration uses two inverting amplifiers, which we can also rule out. Inverting our negative input twice will not produce a positive 10V output as desired.

   Both configurations 3 and 4 are acceptable, as long as a sound explanation was given. Each employs one inverting amplifier and one voltage follower, albeit in different orders. Configuration 3 first inverts the signal and then follows it through to create an ideal source to driver the speakers, while configuration 4 first follows the input through to produce an ideal input before inverting it.

   In either case, the expected ratio of resistors was $-\frac{R_2}{R_1} = \frac{10V}{-35mV}$. This is most closely accomplished using $\boxed{R_1 = 10\Omega}$ and $\boxed{R_2 = 3k\Omega}$. Other ratios which were reasonably close were accepted as well.

4 Solution 4

1. The output voltage $V_{out}(t)$ of op-amp circuit 1 reaches steady-state (where the voltage is constant at $t >> 0$) because when the source is disconnected due to an open circuit of the capacitor at steady-state, $V_{out}(t >> 0) = 0$ is constant.

   However, in op-amp configuration 2,

   $$i_L(t) \propto \int_{t=0}^{t=\infty} v_L(t)\,dt \to \infty, \tag{24}$$

   and since the voltage across the capacitor $v_C(t)$ depends on the current through $i_L(t)$ as a result of $i_L(t) = i_C(t)$,

   $$v_C(t) \propto \int_{t=0}^{t=\infty} i_C(t)\,dt \propto \int_{t=0}^{t=\infty} i_L(t)\,dt \to \infty. \tag{25}$$

   Since $v_C(t) = V_{out}(t)$ is not constant, $V_{out}(t)$ does not reach steady-state.

2. Given that a sinsoidal source is supplied to a circuit that contains a combination of linear elements (resistors, inductors, and capacitors), the total response, or general solution to the linear ordinary differential equation, is in the following form

   $$x(t) = Ae^{-t/\tau} + B\cos(\omega t) + C\sin(\omega t), \tag{26}$$

   where $x(t)$ is the voltage or current quantity in the circuit, τ is the time constant, ω is the frequency, and A, B, C are constants.

   If the sinusoidal source continues to supply energy to the circuit as $t \to \infty$, then from Equation (26) the exponential term goes to zero. The remaining terms can be simplified to the form,

   $$x(t) = K\cos(\omega t + \theta), \tag{27}$$

   where θ is the resulting phase shift and K is a constant term derived from constants B and C.

   By definition of a phasor, the relation of a phasor to a time-varying sinusoid with phase ϕ and amplitude X_m is denoted by

   $$x(t) = X_m cos(\omega t + \phi) \Leftrightarrow X_m e^{Sj\phi} = X_m \angle \phi. \tag{28}$$

   Hence, when a sinusoidal source is supplied to the circuit, the steady-state response of the circuit may be represented by a phasor.

3. Phasors are used to describe the relationship between V_S and V_{out} in terms of the angular frequency ω, for $Z_R = R$, $Z_C = -j(\frac{1}{\omega C})$, and $Z_L = j\omega L$ at $t = 0$.

   Applying KCL at node $\mathbf{V_-}$ for $\mathbf{V_-} = 0$ yields,

$$\frac{\mathbf{V_S} - \mathbf{V_-}}{Z_L} = \frac{\mathbf{V_-} - \mathbf{V_{out}}}{Z_R} + \frac{\mathbf{V_-} - \mathbf{V_{out}}}{Z_C}. \tag{29}$$

   Simplifying the above expression in terms of ω results in

$$\mathbf{V_{out}} = -\frac{1}{Z_L} \cdot \frac{Z_R Z_C}{Z_R + Z_C} \cdot \mathbf{V_S} = \frac{-\mathbf{V_S}}{j\omega L(j\omega R^2 C + R)}. \tag{30}$$

   In phasor form, $\mathbf{V_S} = -V_m \angle\theta$, hence, by substituting into Equation(30), an expression is obtained in phasor form for $\mathbf{V_{out}}$,

$$\mathbf{V_{out}} = \frac{-V_m \angle\theta}{M_{out}\angle\alpha} = \frac{-V_m \angle\theta}{\omega RL(1 + \omega^2 R^2 C^2)^{1/2} \angle\alpha}. \tag{31}$$

   where M_{out} is the magnitude and α is the phase angle of $\mathbf{V_{out}}$.

   Then it follows that by taking the magnitude of $\mathbf{V_{out}}$ yields the following expression,

$$|\mathbf{V_{out}}| = \frac{|-V_m|}{|M_{out}|} = \frac{V_m}{\omega RL(1 + \omega^2 R^2 C^2)^{1/2}}. \tag{32}$$

   Therefore, from Equation(32), it can be shown that as ω increases, the magnitude of V_{out} decreases relative to the magnitude of V_S.

5 Solution 5

(a) Let us call the impedance of the quisistor as Z_Q. Since it has achieved sinusoidal steady-state, without loss of generality, we can assume $v_Q = A\sin(\omega t + \phi)$ where A is the amplitude,ω is the frequency and ϕ is the phase.In order to calculate the complex impedance,let us first differentiate v_Q twice in order to be able to find the current through the quisistor:

$$\begin{aligned} \frac{d^2 v_Q(t)}{dt^2} &= -A\omega^2 \sin(\omega t + \phi) && (33) \\ i_Q(t) &= -Q\frac{d^2 v_Q(t)}{dt^2} && (34) \\ i_Q(t) &= -Q\{-A\omega^2 \sin(\omega t + \phi)\} && (35) \\ i_Q(t) &= QA\omega^2 \sin(\omega t + \phi) && (36) \end{aligned}$$

Now, that we've found,$v_Q(t)$ and $i_Q(t)$, we can find Z_Q by taking their ratios.

$$\begin{aligned} Z_Q &= \frac{v_Q(t)}{i_Q(t)} && (37)\\ &= \frac{A\sin(\omega t+\phi)}{QA\omega^2\sin(\omega t+\phi)} && (38)\\ Z_Q &= \frac{1}{Q\omega^2} && (39) \end{aligned}$$

Notice that the impedance is a purely real number. Therefore,

$$\boxed{Z_Q = \frac{1}{Q\omega^2}\Omega}$$

(b) i. Apply Kirchoff's Voltage law and then substitute for the current to get the final differential equation:

$$\begin{aligned} V_s(t) &= Ri_Q(t) + v_Q(t) && (40)\\ V_s(t) &= R\{-Q\frac{d^2v_Q(t)}{dt^2}\} + v_Q(t) && (41)\\ \frac{-V_s(t)}{RQ} &= \frac{d^2v_Q(t)}{dt^2} + \frac{-v_Q(t)}{RQ} && (42) \end{aligned}$$

The final form should be as follows :

$$\boxed{\frac{-V_s(t)}{RQ} = \frac{d^2v_Q(t)}{dt^2} + \frac{-v_Q(t)}{RQ}}$$

ii. In order to find a relationship between V_s and v_Q (ignoring the t variable), let us apply Kirchoff's Current Law at node a .

$$\begin{aligned} \frac{v_Q - V_s}{R} + \frac{v_Q - 0}{Z_Q} &= 0 && (43)\\ v_Q - V_s + v_QRQ\omega^2 &= 0 && (44)\\ v_Q(1+RQ\omega^2) &= V_s && (45)\\ v_Q &= \frac{V_s}{1+RQ\omega^2} && (46) \end{aligned}$$

On substituing the values of $R = 1; Q = 2; \omega = 2$, we get

$$\boxed{v_Q(t) = \frac{V_s(t)}{9} = \frac{2}{9}\cos(2t+3)V}$$

iii. Steady-state is not reached for $v_Q(t)$, given the initial conditions. Because we initially have $V_S(t) > v_Q(t)$, positive current flows across the resistor into the quisistor. According to the given quisistor relation, we can see that the second derivative of voltage is thus negative. What does this mean for $v_Q(t)$?

Mathematically, a negative second derivative means that the rate of change is decreasing; graphically, it also means that the function's shape is concave-downward. A quick check by integrating $i_Q(t)$ twice would also give you that conclusion. As $i_Q(t)$ is initially constant, we have

$$\int -Cdt = -Ct$$
$$\int -Ctdt = -Dt^2$$

So $v_Q(t)$ decreases quadratically with time.

As $v_Q(t)$ decreases, the voltage drop across the resistor correspondingly increases. By Ohm's Law, more current will be drawn across the resistor and into the quisistor. This increasing current will only serve to decrease $v_Q(t)$ even faster, thus producing $i_Q(t) \to \infty$ and $v_Q(t) \to -\infty$ as $t \to \infty$.

iv. Recall that power $= P = IV$, where I is the current flowing into the positive reference terminal of the voltage V. For a power supplier, the power is negative (current flows out of the positive reference voltage terminal), and for a power absorber, the power is positive (current flows into the positive reference voltage terminal).

The current-voltage relationship for the quisistor is such that as the voltage across the quisistor accelerates (becomes more positive), the current is negative, and hence flows out of the positive reference voltage terminal. If the reference voltage is already positive, and continues to accelerate and becomes even more positive, then current flows out of the positive reference voltage terminal, implying that the quisistor is now a voltage supplier. However, if the reference voltage is positive, and is decelerating (becoming more negative), the current is positive, and hence flows into the positive reference terminal. At this point, the voltage may still be positive (we can't tell if the voltage is positive or negative by how much it accelerates), in which case the quisistor becomes a power absorber.

Hence, the quisistor can be both a power supplier and a power absorber.