Solution to MT 2: Problem#1

(a) Streamlines passing through A=(1,1): $\Psi(1,1) = V_0 l \frac{x}{l}\frac{y}{l} = V_0 l$

=> equation of the streamline is the hyperbola: $\frac{y}{l} = 1/\frac{x}{l}$

Streamlines passing through B=(3,2): $\Psi(3,2) = V_0 l \frac{x}{l}\frac{y}{l} = 6V_0 l$

=> equation of the streamline is: $\frac{y}{l} = 6/(x/l)$

(b) Flow rate through a line across points A and B: $Q_{AB} = \Psi_B - \Psi_A = \Psi(3,2) - \Psi(1,1) = 5V_0 l$

This flow rate is per unit z, so it has the dimension L^2/T.

(c) Material acceleration (use material derivative):

$$a_x = \frac{\partial u}{\partial t} + (\vec{V}.\nabla)u = \frac{x}{l}(\dot{V}_0 + V_0^2/l)$$

and

$$a_y = \frac{\partial v}{\partial t} + (\vec{V}.\nabla)v = \frac{y}{l}(-\dot{V}_0 + V_0^2/l)$$

One contribution is the rate of change of V_0 with time; the second contribution is from the convective derivative, change following the velocity vector:

Velocity vector: $\vec{V} = u\hat{e}_x + v\hat{e}_y = \frac{x}{l}V_0\hat{e}_x - \frac{y}{l}V_0\hat{e}_y$

Radial vector: . $\vec{r} = \frac{x}{l}\hat{e}_x + \frac{y}{l}\hat{e}_y$

Comparing the acceleration expressions with the above vectors:

$$\vec{a} = a_x\hat{e}_x + a_y\hat{e}_y = \dot{V}_0(\frac{x}{l}\hat{e}_x - \frac{y}{l}\hat{e}_y) + V_0^2/l.(\frac{x}{l}\hat{e}_x + \frac{y}{l}\hat{e}_y)$$

$$\vec{a} = \dot{V}_0/V_0(\frac{x}{l}V_0\hat{e}_x - \frac{y}{l}V_0\hat{e}_y) + V_0^2/l.(\frac{x}{l}\hat{e}_x + \frac{y}{l}\hat{e}_y)$$

One component lines up with vector V, the other lines up With the radial vector from the origin.

$$\boxed{\vec{a} = \left(\dot{V}_0/V_0\right)\vec{V} + \left(V_0^2/l\right)\vec{r}}$$

(d) To make the material acceleration in the y-direction vanish, we need:

$\frac{y}{l}(-\dot{V}_0 + V_0^2/l) = 0$ for all y, which is possible. Hence the condition: $\boxed{\dot{V}_0 = V_0^2/l}$

However, the x-acceleration cannot be made to vanish.

(e) The differential equation satisfied by V_0 is: $(\dot{V}_0 - V_0^2/l) = 0$

This equation can be solved separating the V_o & t variables: . $\frac{dV_0}{V_0^2} = \frac{dt}{l}$

Integrating both sides and using the initial condition $V_0(0)=K$, we get: $\frac{-1}{V_0} - \frac{-1}{K} = \frac{t}{l}$

or $\boxed{V_0(t)/K = 1/(1 - \frac{Kt}{l})}$

Solution to MT2: Problem#2

(a) Mass conservation: $\rho A_0 v_0 = \rho A_1 v_1$ => $v_1 = \frac{A_0}{A_1} v_0$ (1)

(b) Bernoulli's equation is applicable for an incompressible and inviscid fluid, **in a steady flow**, along a streamline. We assume that it can be used even when the flow is unsteady. We take z_1=0.

$$\frac{p_0}{\rho} + \tfrac{1}{2} v_0^2 + g z_0 = \frac{p_1}{\rho} + \tfrac{1}{2} v_1^2 + g z_1$$

In the present case: $p_0 = p_1 = p_{atm}$ => $\tfrac{1}{2}(v_1^2 - v_0^2) = g(z_0 - z_1)$ (2)

At t~0, we can assume that v_0~0 and (z_0-z_1) ~ h_0 => $v_1 \approx \sqrt{2 g h_0}$

(c) Assuming a constant exiting velocity v_1, then it is straight forward to esimate:

$$\boxed{T_A = A_0 h_0 / A_1 v_1 = A_0 h_0 / A_1 \sqrt{2 g h_0}}$$

$$T_A = \frac{A_0}{A_1}\sqrt{\frac{h_0}{2g}} = \frac{10 m^2}{0.2 m^2}\sqrt{\frac{3m}{2 \times 9.81 m\, s^{-2}}} \approx 19.6 \sec$$

(d) Equation (2), with *h(t)* now being variable, becomes: . $\tfrac{1}{2}(v_1^2 - v_0^2) = g h(t)$

(e) Combining this equation with equation (1), we obtain: . $v_0^2 = 2 g h(t) / [A_0^2 / A_1^2 - 1]$

Noting that : $\frac{dh}{dt} = -v_0$

$$v_0 = \sqrt{2 g h(t) / [A_0^2 / A_1^2 - 1]} \quad \text{or} \quad \boxed{\frac{dh}{dt} = -h^{1/2}(t)\sqrt{2g / [A_0^2 / A_1^2 - 1]}}$$

This is a 1st order ordinary differential equation in *h(t)*, with the radical term being just a constant.

(f) **Extra credit.** Solving the above ODE by separating the *h* and *t* variables yields:

$$\frac{dh}{\sqrt{h(t)}} = -\sqrt{2g / [A_0^2 / A_1^2 - 1]}\, dt \quad => \quad 2\sqrt{h(t)}\Big|_{h_0}^{h(t)} = -\sqrt{2g / [A_0^2 / A_1^2 - 1]}\; t$$

The drain time T_B is such that $h(t=T_B) = 0$:

$$2\sqrt{h_0} = \sqrt{2g / [A_0^2 / A_1^2 - 1]}\, T_B \quad => \quad \boxed{T_B = 2\sqrt{[A_0^2 / A_1^2 - 1] / 2g}\sqrt{h_0} \approx 39.1 s}$$

Solution to MT2 - Problem #3 (Using velocity not flow rate):

(a) The CV is the part of fluid inside the pipe, then cut off at inlet (2) and at outlet (2)

Using Conservation of Mass Equation, we can relate the velocities at the inlet (1) and outlet (2):

$$v_1 A_1 = v_2 A_2 \quad \Rightarrow \frac{v_2}{v_1} = \frac{A_1}{A_2} = \frac{D_1^2}{D_2^2} = \frac{1}{0.7^2} \cong 2 \qquad (1)$$

Using Bernoulli's Equation with no loss, we can relate the pressures at the inlet(1) and outlet(2):

$$\frac{v_1^2}{2} + \frac{p_1}{\rho} = \frac{v_2^2}{2} + \frac{p_2}{\rho} \quad \Rightarrow p_2 = \frac{\rho v_1^2}{2}\left(1 - \left(\frac{v_2}{v_1}\right)^2\right) + p_1$$

This gives the pressure ratio:

$$\boxed{\frac{p_2}{p_1} = 1 + \frac{\rho v_1^2}{2p_1}\left(1 - \left(\frac{v_2}{v_1}\right)^2\right) = 1 + \frac{\rho v_1^2}{2p_1}\left(1 - \frac{1}{0.7^4}\right) \cong 1 - \frac{3\rho v_1^2}{2p_1} \qquad (2)}$$

The pressure-velocity "parameter" $p_1/\rho v_1^2$ shows up **naturally**, as in the "bent pipe example" in class.

(b) Steady flow, no time derivative term, RTT for momentum in the x-direction:

$$-v_1 \rho v_1 A_1 = p_1 A_1 + F_x \quad , \quad \Rightarrow -\left(v_1^2 \rho + p_1\right) A_1 = F_x, \quad or \quad \frac{F_x}{\rho v^2{}_1 A_1} = -\left(1 + \frac{p_1}{\rho v^2{}_1}\right), \text{ (Note} < 0.)$$

Note that the velocities are all related to Q_1 $\quad v_1 = \frac{4Q_1}{\pi D_1^2} = 0.49 v_2$

(c) The equation for conservation of linear momentum applied in the y-direction:

$$v_2 \rho v_2 A_2 = -p_2 A_2 + F_y \qquad \left(v_2^2 \rho + p_2\right) A_2 = F_y \qquad \boxed{\frac{F_y}{\rho v_2^2 A_2} = \left(1 + \frac{p_2}{\rho v_2^2}\right) \quad (3)}$$

The above expression can be made comparable to the *Fx* expression if we get an expression for $p_2/\rho v_2^2$. This can be simply obtained by dividing (2) by $\rho v_2^2/p_1$:

$$\frac{p_2}{\rho v_2^2} = \frac{p_1}{\rho v_2^2} + \frac{1}{2}\left(\left(\frac{v_1}{v_2}\right)^2 - 1\right) = \frac{0.7^4 p_1}{\rho v_1^2} + \frac{1}{2}\left(\left(0.7^2\right)^2 - 1\right) = 0.7^4\left(\frac{p_1}{\rho v_1^2} + \frac{1}{2}\left(1 - \frac{1}{0.7^4}\right)\right)$$

-1.6

This gives the final expression for Fy in terms of the inlet variables:

$$\frac{F_y}{\rho v_1^2 A_1} = \frac{F_y}{\rho v_2^2 A_2}\frac{A_2}{A_1}\left(\frac{v_2}{v_1}\right)^2 = \frac{F_y}{\rho v_2^2 A_2}\frac{1}{0.7^2} = \boxed{\frac{1}{0.7^2}\left(1 + 0.7^4\left(\frac{p_1}{\rho v_1^2} - 1.6\right)\right)} \qquad (4)$$

(d) Now that the Fx and Fy components due to the walls have been found the angle with which they act is:

$$\theta = \tan^{-1}\left[\frac{\frac{F_y}{\rho v_1^2 A_1}}{\frac{F_x}{\rho v_1^2 A_1}}\right] - \frac{\pi}{2} \cong \tan^{-1}\left[\frac{1.265 + \frac{0.49 p_1}{\rho v_1^2}}{-(1 + \frac{p_1}{\rho v_1^2})}\right] - \frac{\pi}{2}$$

No numerical values were given, thus the exact angle cannot be determined. However, the range that the angle must be as follows (measured counterclockwise from +y-axis to be positive) as Fx is <0): $\frac{\pi}{2} < \theta < 0$