MT3 Solutions (Phys 112 S09)

1. Black body radiation pressure (15 pts)

a. (5 pts) Show $\mathcal{D}(\varepsilon)d\varepsilon = \dfrac{\varepsilon^2 d\varepsilon}{\pi^2 \hbar^3 c^3}$

$$\underbrace{2}_{\text{2 indp. polarizations for photons}} \times \int_{\text{Space}} \int_{\Omega_p} \left(\frac{1}{h^3}\right) d^3x\, d^3p = 2 \times \frac{4\pi V}{h^3} p^2 dp$$

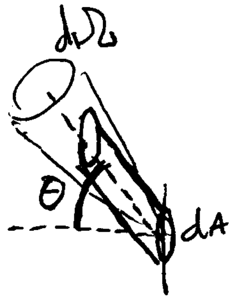
$$= 2 \times \frac{4\pi V}{h^3}\left(\frac{\varepsilon^2}{c^2}\right)\left(\frac{d\varepsilon}{c}\right)$$

($\varepsilon = pc$ for relativistic particles)

$$= V \times \frac{8\pi}{8\pi^3 \hbar^3} \frac{\varepsilon^2 d\varepsilon}{c^3} = V \times \frac{\varepsilon^2 d\varepsilon}{\pi^2 \hbar^3 c^3}$$

$$= V \times \mathcal{D}(\varepsilon)\, d\varepsilon \quad \checkmark$$

b. (5 pts)



$$\text{photon flux} = \underbrace{\overbrace{\left[\frac{c\,dt}{dt} dA \cos\theta\right]}^{\text{(volume incident)}}}_{\text{per unit time}} \overbrace{\left[\mathcal{D}(\varepsilon)\, d\varepsilon\right]}^{\text{(\# states per volume)}} \overbrace{\left[\frac{1}{e^{\varepsilon/\tau} - 1}\right]}^{\text{(\# photons in each state)}} \underbrace{\frac{d\Omega}{4\pi}}_{\text{per solid angle}}$$

$$= c\, dA \cos\theta \, \frac{\varepsilon^2 d\varepsilon}{\pi^2 \hbar^3 c^3} \, \frac{1}{e^{\varepsilon/\tau} - 1} \, \frac{d\Omega}{4\pi}$$

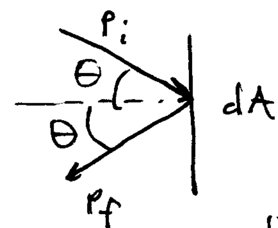
$$= \frac{1}{e^{\varepsilon/\tau} - 1} \cos\theta \, \frac{\varepsilon^2}{4\pi^3 \hbar^3 c^2} \, d\varepsilon\, d\Omega\, dA \quad \checkmark$$

c. (5 pts)

(flux of photons) × (momentum of each photon delivered)

= force per unit area per solid angle per energy

momentum delivered:



$\Delta p = 2p\cos\theta$

$|\vec{p}_i| = |\vec{p}_f|$

$\Delta p = \dfrac{2\varepsilon}{c}\cos\theta$

$$\Rightarrow dF = \frac{1}{e^{\varepsilon/T}-1}\cos\theta\,\frac{\varepsilon^2}{4\pi^3\hbar^3c^2}\,d\varepsilon\,(\sin\theta\,d\theta\,d\phi)\,dA \times \frac{2\varepsilon}{c}\cos\theta$$

$$= \frac{1}{2\pi}\underline{\underline{\cos^2\theta\sin\theta\,d\theta\,d\phi}}\;u_\varepsilon\,d\varepsilon\,dA$$

$\left(u_\varepsilon = \dfrac{u_\omega}{\hbar}\right)$

Integrate over all angles θ, ϕ:

only $\pi/2$, not π, because only incoming photons

$$\int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\;\frac{1}{2\pi}\cos^2\theta\sin\theta\,d\theta\,d\phi \times u_\varepsilon\,d\varepsilon\,dA$$

$$= \frac{1}{3}\cos^3\theta\Big|_{\pi/2}^{0}\;u_\varepsilon\,d\varepsilon\,dA$$

$$= \frac{1}{3}u_\varepsilon\,d\varepsilon\,dA$$

integrate over all energies:

$$\frac{1}{3}\frac{U}{V}\,dA = dF \Rightarrow \underline{\underline{P = \frac{dF}{dA} = \frac{1}{3}\frac{U}{V}}}$$

So the radiation pressure is $\frac{1}{3}$ the spatial energy density.

2. Greenhouse effect (20 pts)

a. $\begin{pmatrix}\text{power received}\\ \text{by Earth}\end{pmatrix} = \begin{pmatrix}\text{power emitted}\\ \text{by Earth}\end{pmatrix}$

$$\frac{P_\odot}{4\pi D_{SE}^2}\pi R_E^2 = P_\oplus = \sigma_B 4\pi R_E^2 T_E^4$$

$$\sigma_B 4\pi R_S^2 T_S^4 \times \frac{\pi R_E^2}{4\pi D_{SE}^2} = \sigma_B 4\pi R_E^2 T_E^4$$

$$\Rightarrow \underline{\underline{T_E = \sqrt{\frac{R_S}{2D_{SE}}}\, T_S}} \approx \underline{\underline{280K}}$$

b. $\underline{P_E = \frac{1}{2}P_{GH} + P_S}$

$P_E = \sigma_B 4\pi R_E^2 T_E^4$, $\quad P_{GH} = 2\sigma_B 4\pi R_E^2 T_{GH}^4$

$P_S = \sigma_B 4\pi R_S^2 T_S^4 \dfrac{\pi R_E^2}{4\pi D_{SE}^2}$

$P_{GH} = P_E$, so $P_E = \frac{1}{2}P_E + P_S$

or $P_S = \frac{1}{2}P_E$, so now we get $\underline{\underline{T_E^{(new)} = 2^{1/4}\, T_E^{(old)}}}$

$2^{1/4} \simeq 1.18$, so $\underline{\underline{T_E^{(new)} \simeq 333\text{ K}}}$

3. a. (5 pts) phase space density of states is

$$2 \times \frac{1}{h^2}\, d^2x\, d^2p$$

density in 2D, e^- is spin ½, so ×2.

b. $\mathcal{D}(\varepsilon)\,d\varepsilon$?

$$\frac{2}{h^2} \int_{\text{space}} \int_{\substack{\text{momentum}\\ \text{angles}}} d^2x \; p\,dp\,d\theta = \frac{2A}{h^2}\, 2\pi\, p\,dp$$

$$= \frac{4\pi A}{4\pi^2 \hbar^2} \sqrt{2m\varepsilon}\; \frac{\sqrt{2m}}{2}\, \varepsilon^{-1/2}\, d\varepsilon = A \times \frac{m}{\pi\hbar^2}\, d\varepsilon$$

So $\mathcal{D}(\varepsilon)\,d\varepsilon = \frac{m}{\pi\hbar^2}\, d\varepsilon$

c. $f_{FD} \to 1$ for $\varepsilon < \mu$ @ 0 temp.

$$\langle N \rangle = \int_0^\infty d\varepsilon\; \mathcal{D}(\varepsilon)\, f_{FD}(\varepsilon) \xrightarrow{T \to 0} \int_0^{\mu = \varepsilon_F} d\varepsilon\, \frac{Am}{\pi\hbar^2} \times 1$$

$$= \frac{Am}{\pi\hbar^2}\, \mu$$

$$\mu = \frac{(\langle N \rangle / A)\, \pi\hbar^2}{m} = \frac{\pi\hbar^2}{m}\, n$$

$$\boxed{\mu = \frac{\pi\hbar^2}{m}\, n}$$ @ 0 temperature.