Engineering 117
Fall Semester 2010
First Midterm Examination

October 8, 2010

SEVENTY-FIVE MINUTES, CLOSED BOOK. ONE: $8\frac{1}{2}'' \times 11''$ SHEET OF NOTES ALLOWED.

1. A differential equation for a function $y(x)$ is given by

$$y''(x) - 2y'(x) + y(x) = 8\ \cosh(x)$$

   (a) How many initial conditions are required to find the complete solution for $y(x)$?

   Two.

   (b) How many independent solutions to the homogeneous equation $y''(x) - 2y'(x) + y(x) = 0$ must be found? What are these solutions?

   Two. Since the characteristic equation $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$ has a double root at $\lambda = 1$, the two solutions are e^x and xe^x.

   (c) Does the right hand side of the equation contain terms which are a solution to the homogeneous equation ? If so, what would you take as a form $y_p(x)$ for the particular response? (Hint: write out cosh in terms of elementary exponential functions.)

   Since $\cosh x = 1/2(e^x + e^{-x})$, the forcing term has a component which is part of the homogeneous response, on its double root. Therefore the particular response is of the form $Ax^2e^x + Be^{-x}$.

   (d) Solve for for $y_p(x)$.

   Solving for $\mathcal{L}\left[Ax^2e^x\right] = 4e^x$ and $\mathcal{L}\left[Be^{-x}\right] = 4e^{-x}$ gives $A = 2$ and $B = 1$, thus $y_p(x) = 2x^2e^x + e^{-x}$.

   (e) For initial conditions $y(0) = 0$ and $y'(0) = -1$, solve for the complete response. **Note that $y_p(0) = 1$ and $y_p'(0) = -1$. So**

$\mathbf{y_h(0) = -1}$ **and** $\mathbf{y_h'(0) = 0}$**. Since** $\mathbf{y_h(x) = Ce^x + Dxe^x}$**, this is satisfied when** $\mathbf{C = -1}$ **and** $\mathbf{D = 1}$**, thus:**

$$\mathbf{y(x) = 2x^2e^x + e^{-x} - e^x + xe^x}$$

2. A system of ODEs is written in the form

$$\frac{d\mathbf{y}(t)}{dt} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{y}(t)$$

(a) Solve for the eigenvalues (λ_1, λ_2) of the matrix operating on $\mathbf{y}(t)$ on the RHS and then write down a general solution for $\mathbf{y}(t)$ in terms of two independent vectors $\mathbf{y}_1$ and $\mathbf{y}_2$, yet to be determined.
$\det(\lambda \mathbf{I} - \mathbf{A}) = \boldsymbol{\lambda^2 + 3\lambda + 1 = (\lambda + 3)(\lambda + 1)}$ **which gives** $\boxed{\boldsymbol{(\lambda_1, \lambda_2) = (-3, -1).}}$ **Then** $\boxed{\mathbf{y(t) = y_1e^{-3t} + y_2e^{-t}}}$

(b) Now solve for the two vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ by using the eigenvalues calculated in part (a).
For $\boldsymbol{\lambda = -3}$**, we seek the solution to**

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is satisfied for

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\boldsymbol{\lambda = -1}$**, we seek the solution to**

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is satisfied for

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and thus the general solution is

$$\mathbf{y(t) = c_1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathbf{e^{-3t} + c_2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{e^{-t}}$$

(c) If the initial condition is $\mathbf{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find $\mathbf{y}(t)$ for all time $t > 0$.
By inspection, $\mathbf{(c_1, c_2) = (-1/2, 1/2)}$ **and thus**

$$\mathbf{y(t)} = \frac{\mathbf{-1}}{\mathbf{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathbf{e^{-3t}} + \frac{\mathbf{1}}{\mathbf{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{e^{-t}}$$

3. A differential equation for a forced system is given by

$$y''(t) + 3y'(t) + 2y(t) = f(t)$$

where

$$f(t) = \begin{cases} \cos t - 3\sin(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

(a) Write down a Laplace transform equation in the form $\mathbf{G}(s)\mathbf{Y}(s) + \mathbf{IC}(s) = \mathbf{F}(s)$, where $\mathbf{GY} + \mathbf{IC}$ and $\mathbf{F}$ are the transforms of the LHS and RHS, respectively. Leave the initial conditions $\mathbf{IC}$ as arbitrary at this point.

$$\left(\mathbf{s^2 + 3s + 2}\right)\mathbf{F(s) - (s+3)f(0) - f'(0) = \frac{s-3}{s^2+1}}$$

(b) For initial conditions $y(0) = 1$ and $y'(0) = 0$, solve for $\mathbf{Y}(s)$. Write this as an expanded partial fraction.

$$\begin{aligned} Y(s) &= \left(\frac{1}{(s+2)(s+1)}\right)\left(3 + s + \frac{s-3}{s^2+1}\right) \\ &= \frac{s(s+2)(s+1)}{(s+2)(s+1)(s^2+1)} \\ &= \frac{s}{s^2+1} \end{aligned}$$

(c) Find the complete solution $y(t)$ for all times $t > 0$.

$$\boxed{\mathbf{y(t) = \cos(t)}}$$