Math 1B Final Exam
Friday, 15 August 2008

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Name: *Answers*

Problem Number	1	2	3	4	5	6	7	Total
Score								
Maximum	10	20	10	15	15	15	15	100

Please do not begin this test until 8:10 a.m. The test ends exactly at 10 a.m.
As always, show work for partial credit. Please box your final answers.

1. (10 pts – 5 questions, 2 pts each) Determine whether the following statements are true or false. Full points will be awarded for the correct answer; partial credit may be awarded for useful thoughts without the correct answer. Throughout, a_n, b_n, and c_n are unknown sequences of (possibly negative) real numbers.

   (a) TRUE or FALSE: If $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = 0$ and $\sum_{n=1}^{\infty} b_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

   TRUE. This is a restatement of one part of the limit-comparison test.

   (b) TRUE or FALSE: If $\sum_{n=1}^{\infty} a_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n$ converges conditionally, then $\sum_{n=1}^{\infty} (a_n + b_n)$ converges conditionally.

   TRUE. The sum of convergent series is convergent, but the difference of absolutely convergent series is absolutely convergent, so this sum must be conditionally convergent.

   (c) TRUE or FALSE: If $\sum_{n=1}^{\infty} c_n(-4)^n$ converges, then $\sum_{n=1}^{\infty} c_n 3^n$ converges absolutely.

   TRUE. The radius of convergence of $\sum c_n x^n$ must be at least $|-4| = 4$, and power series converge absolutely inside their radii.

   (d) TRUE or FALSE: If $a_n \leq b_n$ for every n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   FALSE. The Comparison Test requires that all terms in the series be positive.

   (e) TRUE or FALSE: If $\lim_{n\to\infty} [a_{n+1} - a_n] \neq 0$, then $\lim_{n\to\infty} a_n$ does not converge.

   TRUE. This is a restatement of the Divergence Test.

2. (20 pts – 4 questions, 5 pts each) Determine whether the following series converge absolutely, converge conditionally, or diverge. Explain how you know.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n + \frac{1}{n+1}}$

By limit-comparison, this series behaves like the alternating harmonic series, so must $\boxed{\textit{converge conditionally}}$*. More details: it satisfies the conditions of the alternating series test, since* $\dfrac{1}{n+1} \leq \dfrac{1}{n + \frac{1}{n+1}} \leq \dfrac{1}{n}$*, so converges, but by comparison with the harmonic series, diverges in absolute value.*

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n \arctan n}{3^n}$

Since $|\arctan n| \leq \pi/2$*, we see that* $\left|\dfrac{(-1)^n}{n + \frac{1}{n+1}}\right| \leq \dfrac{\pi}{2}\left(\dfrac{2}{3}\right)^n$*, so the series* $\boxed{\textit{converges absolutely}}$ *by comparison test with the geometric series* $\sum(2/3)^n$*.*

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n(n+2)}{(n+3)^2}$

The series *diverges* *by the divergence test:*

$$\lim_{n\to\infty}\frac{n(n+2)}{(n+3)^2} = 1 \text{ so } \lim_{n\to\infty}(-1)^n\frac{n(n+2)}{(n+3)^2} = \text{DNE}.$$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\overbrace{2\cdot 5\cdot 8\cdot\ldots\cdot(3n-4)\cdot(3n-1)}^{n\text{ numbers}}}{\underbrace{2\cdot 6\cdot 12\cdot 20\cdot 30\cdot\ldots\cdot(n-1)n\cdot n(n+1)}_{n\text{ numbers}}}$

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\dfrac{\overbrace{2\cdot 5\cdot 8\cdot\ldots\cdot(3n-1)\cdot(3n+2)}^{n+1\text{ numbers}}}{\underbrace{2\cdot 6\cdot 12\cdot 20\cdot 30\cdot\ldots\cdot n(n+1)\cdot(n+1)(n+2)}_{n+1\text{ numbers}}}}{\dfrac{\overbrace{2\cdot 5\cdot 8\cdot\ldots\cdot(3n-4)\cdot(3n-1)}^{n\text{ numbers}}}{\underbrace{2\cdot 6\cdot 12\cdot 20\cdot 30\cdot\ldots\cdot(n-1)n\cdot n(n+1)}_{n\text{ numbers}}}}$$

$$= \lim_{n\to\infty}\frac{3n+2}{(n+1)(n+2)}$$

$$= 0 < 1$$

So the series *converges absolutely*.

3. (10 pts) Find the radius and interval of convergence of the following power series.

$$\sum_{n=0}^{\infty} \frac{(n+1)}{4^n(n+2)^2}(x-2)^n$$

We begin with the ratio test to compute the radius of convergence:

$$\begin{aligned} R.O.C. &= \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| \\ &= \lim_{n\to\infty} \frac{(n+1)}{4^n(n+2)^2} \frac{4^{n+1}(n+3)^2}{(n+2)} \\ &= \lim_{n\to\infty} 4 \cdot \frac{(n+1)(n+3)^2}{(n+2)^3} \\ &= \boxed{4} \end{aligned}$$

So we have a power series centered at 2 *with R.O.C.*= 4*. Thus the endpoints are* -2 *and* 6*. Checking these, we see that at* $x = -2$*, the series is*

$$\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)}{(n+2)^2}$$

which $\boxed{\textit{converges}}$ *by the alternating series test. At* $x = 6$*, the series is*

$$\sum_{n=0}^{\infty} \frac{(n+1)}{(n+2)^2}$$

which $\boxed{\textit{diverges}}$ *by limit-comparison with the harmonic series (equivalently, by the p-test with* $p = 1$*). Thus, the interval of convergence is* $\boxed{x \in [-2, 6)}$.

4. (a) (10 pts) Find a power-series representation centered at $x = 4$ for the function $\sqrt{x}$. You may use any method you wish: manipulating known power series, Taylor's theorem, etc.

If we manipulate known series, we get

$$\begin{aligned}
\sqrt{x} &= \sqrt{4 + (x-4)} \\
&= 2\sqrt{1 + \frac{x-4}{4}} \\
&= 2\sum_{n=0}^{\infty} \binom{1/2}{n} \left(\frac{x-4}{4}\right)^n \\
&= 2 + 2\sum_{n=1}^{\infty} \frac{\overbrace{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\cdots\left(\frac{1}{2} - n + 1\right)}^{n \textit{ numbers}}}{n!\,4^n}(x-4)^n \\
&= 2 + 2\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\overbrace{1 \cdot 3 \cdot \ldots \cdot (2n-3)}^{n-1 \textit{ numbers}}}{n!\,2^n\,4^n}(x-4)^n
\end{aligned}$$

Alternately, we can make a table of derivatives.

n	$\frac{d^n}{dx^n}\sqrt{x}$	$c_n = f^{(n)}(4)/n!$
0	$\sqrt{x}$	2
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}2^{-1}/1$
2	$\frac{1}{2}\frac{-1}{2}x^{-3/2}$	$\frac{-1}{2^2}2^{-3}/2$
3	$\frac{1}{2}\frac{-1}{2}\frac{-3}{2}x^{-5/2}$	$\frac{1 \cdot 3}{2^3}2^{-5}/6$
$\vdots$		$\vdots$
n	$\ldots$	$(-1)^{n-1}\frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{2^{n+2n-1}}/n!$

Using either method we get the same answer:

$$\sqrt{x} = \boxed{2 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{2^{3n-1}\,n!}(x-4)^n}$$

(b) (5 pts) What is the radius of convergence of your answer to part (a)? (You do not need to decide if the series converges at the endpoints.)

By manipulating series, we know that the radius of convergence is $\left|\frac{x-4}{4}\right| < 1$, *i.e.* $\boxed{R.O.C. = 4}$. *Alternately, we can use the ratio test:*

$$\left|\frac{c_n}{c_{n+1}}\right| = \frac{2^3(n+1)}{2n-1} \xrightarrow[n\to\infty]{} \boxed{4}$$

5. (a) (5 pts) Find a power-series representation centered at $x = 0$ for the function $\sin(x/5)$. You may use any method you wish: manipulating known power series, Taylor's theorem, etc.

The even derivatives of $\sin(x/5)$ *at* $x = 0$ *are* 0. *The* $(2k+1)$*st derivative is* $(-1)^k/5^{2k+1}$. *Alternately, we manipulate* $\sin x = \sum_{k=0}^{\infty}(-1)^k x^{2k+1}/(2k+1)!$. *In either case, we get*

$$\sin(x/5) = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{5^{2k+1}(2k+1)!} x^{2k+1}} = 0 + \frac{x}{5} + 0 - \frac{x^3}{5^3 3!} + 0 + \ldots$$

(b) (10 pts) For what n does the nth Taylor polynomial correctly estimate $\sin(2/5)$ to within an error of 0.001? Compute the first three digits of $\sin(2/5)$.

Let's guess that the 4*th polynomial* $0 + x/5 + 0 - x^3/5^3 3! + 0$ *works. It should: the next remainder is at most*

$$|R_4(2)| \leq \frac{2^5}{5^5\, 5!} \leq \frac{(0.4)^5}{100} = .0001024 < .001$$

Thus, we compute

$$\sin(2/5) \approx (0.4) - \frac{(0.4)^3}{3!} = 0.4 - \frac{0.064}{6} = 0.4 - 0.010666\ldots = 0.389333\ldots \approx \boxed{0.389}$$

6. (15 pts) Solve the following initial value problem, by assuming that the solution can be represented by a power series.

$$xy'' + y' - xy = 0, \quad y(0) = 1, \ y'(0) = 0$$

We let $y = \sum_{n=0}^{\infty} c_n x^n$ *and solve for* n*:*

$$\begin{aligned}
xy &= \sum_{n=1}^{\infty} c_{n-1} x^n \\
y' &= \sum_{n=0}^{\infty} c_{n+1}(n+1) x^n \\
xy'' &= \sum_{n=0}^{\infty} c_{n+1} n(n+1) x^n \\
0 = xy'' + y' - xy &= c_1 + \sum_{n=1}^{\infty} \left[(n+1)^2 c_{n+1} - c_{n-1} \right] x^n \\
(n+1)^2 c_{n+1} - c_{n-1} &= 0 \textit{ for } n \geq 1 \\
c_n &= \frac{c_{n-2}}{n^2} \textit{ for } n \geq 2 \\
c_n &= \begin{cases} \frac{c_0}{2^2 \cdot 4^2 \cdot \ldots \cdot n^2}, & n \textit{ even} \\ \frac{c_1}{1^2 \cdot 3^2 \cdot \ldots \cdot n^2}, & n \textit{ odd} \end{cases} \\
c_0 &= 1 \\
c_1 &= 0 \\
c_n &= \begin{cases} \frac{1}{2^2 \cdot 4^2 \cdot \ldots \cdot (2k)^2} = \frac{1}{(2^k k!)^2}, & n = 2k \textit{ even} \\ \frac{0}{1^2 \cdot 3^2 \cdot \ldots \cdot n^2} = 0, & n \textit{ odd} \end{cases} \\
y &= \boxed{\sum_{k=0}^{\infty} \frac{1}{2^{2k}(k!)^2} x^{2k}}
\end{aligned}$$

7. (a) (10 pts) Use the Trapezoid Rule with three subdivisions to estimate $\ln 4 = \int_1^4 \frac{dx}{x}$. What is the expected error of this estimate? Is the estimate too high or too low (hint: draw a picture)? Give a decimal range of possible values for $\ln 4$ based on your estimate.

We have $n = 3$, $a = 1$, $b = 4$, so $\Delta x = 1$. Then

$$T_3 = \left(\frac{1}{2}\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2}\frac{1}{4}\right) \cdot 1 = 0.5 + 0.5 + 0.333\ldots + 0.125 = 1.468\ldots$$

The Trapezoid Rule gives an overestimate of the integral. The maximum second derivative of $1/x$ for $x \in [1, 4]$ is at $x = 1$, where the second derivative is $2/x^3 = 2$. The expected error is at most

$$|E_T| \leq \frac{2(4-1)^3}{12 \cdot 3^2} = 0.5$$

Thus, $\boxed{1.468\ldots \geq \ln 4 \geq (1.468\ldots - 0.5) = 0.968\ldots}$. *(In fact, we know that $\ln 4 > \ln e = 1$.)*

(b) (5 pts) For what n does the Midpoint Rule with n subdivisions estimate $\int_1^4 \frac{dx}{x}$ to within an error of 0.01?

We know that

$$|E_M| \leq \frac{K(b-a)^3}{24\,n^2} = \frac{2 \cdot 3^3}{24\,n^2} = \frac{1}{n^2}\frac{9}{4}$$

We want this to be less than $0.01 = 10^{-2}$. I.e.:

$$\frac{1}{n^2}\frac{9}{4} \leq 10^{-2}$$

$$n^2 \frac{4}{9} \geq 10^2$$

$$n \geq 10\sqrt{\frac{9}{4}} = \boxed{15}$$

so $\boxed{n = 15}$ *works.*

8. (0 pts) Thanks for the great summer! Use this page if you need the extra space.