Midterm 1 Sample Solutions

Problem 1. [True or false] (11 points)

(a) [TRUE] or FALSE: Let $P(x)$ be any polynomial of degree 10 with integer coefficients. Then $P(x)$ has at most 10 roots modulo 19.

(19 is prime, so we can apply fact 1 about polynomials.)

(b) TRUE or [FALSE]: Let $P(x)$ be any polynomial of degree 10 with integer coefficients. Then $P(x)$ has at most 10 roots modulo 20.

(20 is not prime, so fact 1 doesn't apply. For instance, consider the polynomial $P(x) = 10x(x-1)(x-2)\cdots(x-9)$. *Clearly* $0,1,2,\ldots,9$ *are all roots, but 10 is also a root. So there are at least 11 roots even though* $P(x)$ *has degree 10. In fact we can get a sharper counterexample:* $Q(x) = 10x(x-1)$ *has 20 roots but its degree is only 2.)*

(c) [TRUE] or FALSE: There is a unique polynomial $P(x)$ modulo 5 of degree at most 1 such that $P(1) \equiv 2008 \pmod 5$ and $P(2008) \equiv 1 \pmod 5$.

(Reducing mod 5, the listed conditions are equivalent to $P(1) \equiv 3 \pmod 5$ *and* $P(3) \equiv 1 \pmod 5$. *Since 5 is prime, we can apply fact 2 about polynomials.)*

(d) TRUE or [FALSE]: There is a unique polynomial $P(x)$ modulo 5 of degree at most 1 such that $P(3) \equiv 123 \pmod 5$ and $P(123) \equiv 3 \pmod 5$.

(Reducing mod 5, we get only that $P(3) \equiv 3 \pmod 5$. *Thus,* $P(x) = 3$ *and* $P(x) = x$ *both satisfy this condition.)*

(e) [TRUE] or FALSE: Let $f(n)$ denote the maximum number of edges that an undirected graph with n vertices can have. Then $f(n) \in O(n^2)$.

(There can be only 1 edge between every pair of vertices. If we allow self-edges, there are n^2 *possible edges by counting. If we don't, there are less.)*

(f) TRUE or [FALSE]: There are exactly $\binom{10}{3}$ 10-bit strings that contain at most three 1-bits.

($\binom{10}{3}$ *only counts the 10-bit strings that contain exactly 3 1-bits.)*

(g) [TRUE] or FALSE: There are exactly $\frac{52\times51\times50}{3\times2}$ ways to select three cards from a standard deck of 52 cards, if the order in which we select those three cards doesn't matter.

(If order doesn't matter, then there are $\binom{52}{3} = \frac{52\times51\times50}{3\times2}$ *ways to select three cards.)*

(h) TRUE or $\boxed{\text{FALSE}}$: There are exactly $\frac{52\times51\times50}{3\times2}$ ways to select three cards from a standard deck of 52 cards, if the order in which we select those three cards matters.

(If the order matter, then there are 52 permute 3 ways to choose the cards. That's $52\times51\times50$*.)*

(i) TRUE or $\boxed{\text{FALSE}}$: If the events A and B are independent, then it is guaranteed that $\Pr[A\cup B] = \Pr[A] + \Pr[B]$.

($\Pr[A\cup B] = \Pr[A]+\Pr[B]$ *is true iff A,B are disjoint. Independent events do not have to be disjoint.)*

(j) $\boxed{\text{TRUE}}$ or FALSE: If $\Pr[A]\neq 0$ and $\Pr[B]\neq 0$ and $\Pr[A|B] = \Pr[A]$, then it is guaranteed that $\Pr[B|A] = \Pr[B]$.

(By assumption, $\Pr[A] = \Pr[A|B] = \frac{\Pr[A\cap B]}{\Pr[B]}$*, so* $\Pr[A\cap B] = \Pr[A]\times\Pr[B]$*. Then* $\Pr[B|A] = \frac{\Pr[A\cap B]}{\Pr[A]} = \Pr[B]$*.)*

(k) $\boxed{\text{TRUE}}$ or FALSE: If $\Pr[A]\neq 0$ and $\Pr[B]\neq 0$, then it is guaranteed that $\Pr[B|A] = \frac{\Pr[B]}{\Pr[A]}\times\Pr[A|B]$.

(This follows via similar algebraic manipulation as the last problem. In particular, $\Pr[B|A] = \frac{\Pr[A\cap B]}{\Pr[A]} = \frac{\Pr[A|B]\times\Pr[B]}{\Pr[A]} = \frac{\Pr[B]}{\Pr[A]}\times\Pr[A|B]$*.)*

Problem 2. [Proofs and polynomials] (11 points)

Define the sequence $P_1(x), P_2(x),\ldots$ of polynomials as follows: $P_1(x) = x+12$, $P_2(x) = x^2-6x+5$, and

$$P_{n+1}(x) = xP_n(x) - P_{n-1}(x) \qquad \text{for all } n\geq 2.$$

(a) Fill in the blank: $P_1(5) = \boxed{17}$

(b) Fill in the blank: $P_2(5) = \boxed{0}$

(c) Fill in the blank: $P_3(5) = \boxed{5\times P_2(5) - P_1(5) = 5\times 0 - 17 = -17}$

(d) Prove that $P_n(5)\equiv 0 \pmod{17}$ for every integer $n\geq 1$.

Solution: Proof by strong induction over n.

Base cases: If $n=1$, then $P_1(5) = 17\equiv 0 \pmod{17}$ (see part (a)). If $n=2$, then $P_2(5) = 0$ (see part (b)).

Inductive hypothesis: Suppose that $P_j(5)\equiv 0 \pmod{17}$ for $j = 1,2,\ldots,n$.

Inductive step: Suppose $n\geq 2$. Applying the definition from the problem statement and the inductive hypothesis, we find that $P_{n+1}(5) = 5\times P_n(5) - P_{n-1}(5) \equiv 5\times 0 - 0 \equiv 0 \pmod{17}$.

(e) Let q be a prime number. Prove that the polynomial $P_{2008}(x)$ has at most 2008 different roots modulo q.

Solution: First, we prove that the polynomial $P_n(x)$ is of degree n.

Claim: $\deg P_n(x) = n$ for all $n\geq 1$.

Proof: We use proof by strong induction over n.

Base case: If $n=1$, then $P_1(x) = x+12$ has degree 1. If $n=2$, then $P_2(x) = x^2-6x+5$ has degree 2.

Inductive hypothesis: Suppose that $P_j(x)$ has degree j for $j = 1,2,\ldots,n$.

Inductive step: Suppose $n \geq 2$. By the definition of $P_{n+1}(x)$, $P_{n+1}(x) = xP_n(x) - P_{n-1}(x)$. Since $P_n(x)$ has degree n by the inductive hypothesis, $xP_n(x)$ has degree $n+1$. Similarly, $P_{n-1}(x)$ has degree $n-1$. Thus, the coefficient for x^{n+1} term in $P_{n-1}(x)$ is 0 and hence, $P_{n+1}(x) = xP_n(x) + P_{n-1}(x)$ is a polynomial of degree $n+1$. □

Since 17 is prime, we can use fact 1 about polynomials. It follows that $P_n(x)$ has at most n roots for every n.

Alternate solution: We can prove that $\deg P_n(x) \leq n$ for all $n \geq 1$, and then everything else is as before. The proof of this fact is slightly easier, since we don't have to worry about the possibility that subtracting $P_{n-1}(x)$ from $xP_n(x)$ might reduce the degree to less than $\deg xP_n(x)$—if it does, that's not a problem.

Problem 3. [Probability] (18 points)

The card game Euchre is played with a deck of 24 cards consisting of only the 9, 10, J, Q, K, and A of each suit (i.e., we start with an ordinary 52-card deck and remove the cards 2–8). The J, Q, K, and A are considered face cards. There are four players, Debbie, Eve, Frank, and George. In this problem, show your work. You can leave your answer as an unevaluated expression.

(a) Debbie, the dealer, shuffles the deck randomly, so that all orderings are equally likely. How many ways are there to order the deck of 24 cards?

Solution: 24!

(b) After shuffling the deck, Debbie deals five cards to each player. Finally Eve turns up the top card remaining and puts it in the middle of the table. What is the probability that the card Eve turned up is an ace (A)?

Solution: Let A be the event that the card Eve turned up is an ace. The sample space is Ω, the set of all possible orderings of the deck. Since all sample points have uniform probability, $\Pr[A] = \frac{|A|}{|\Omega|} = \frac{4 \cdot 23!}{24!} = \frac{1}{6}$.

Alternate solution: Alternatively, we can consider the sample space to have just 6 sample points, namely the value (not caring about suit) of the top card of the remaining pile. By symmetry, each of these outcome should be equally likely. Hence, the probability of one outcome is just $\frac{1}{6}$.

(c) What is the probability that Frank is dealt a hand with no face cards?

Solution: Let F be the event that Frank is dealt a hand with no face cards.

We can let each sample point be an ordering of the cards. There are 8 non-face cards, so the number of sample points in F is $(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4) \cdot 19!$ since there are 8 ways to choose the first card in Frank's hand, 7 ways to select the second card, and so on. And finally, there are $(24-5)!$ ways select the cards not in Frank's hand. Therefore,

$$\Pr[F] = \frac{|F|}{|\Omega|} = \frac{(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4) \cdot 19!}{24!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}.$$

Alternate solution: We can let each sample point be a possible hand that Frank gets, i.e., a set of cards (ignoring the order in which he got them). Then $|\Omega| = \binom{24}{5}$ and each sample point has uniform probability by symmetry. Thus, $\Pr[F] = \frac{|F|}{|\Omega|} = \frac{\binom{8}{5}}{\binom{24}{5}}$.

Alternate solution: We let the sample space be the sequence of cards that Frank gets, in the order he gets them. Then $|\Omega| = \frac{24!}{19!}$. Then we can define events F_1 = Frank's first card is not a face card, F_2 = Frank's second card is not a face card, and so on. Then,

$$\Pr[F] = \Pr[F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5] = \Pr[F_1]\Pr[F_2|F_1] \cdots \Pr[F_5|F_1 \cap F_2 \cap F_3 \cap F_4] = \frac{8}{24} \cdot \frac{7}{23} \cdot \frac{6}{22} \cdot \frac{5}{21} \cdot \frac{4}{20}$$

by application of the chain rule.

(d) What is the probability that Frank was dealt a hand with no face cards *and* that the card Eve turned up in part (b) was an ace?

Solution: Let the events A, F be as above. Each sample point will represent an ordering of the deck. We need to count the number of sample points in $|A \cap F|$. Since the card Eve turned up is an ace, there are four possibilities for it. Therefore,

$$\Pr[A \cap F] = \frac{|A \cap F|}{|\Omega|} = \frac{4 \cdot (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4) \cdot 18!}{24!} = \frac{4 \cdot (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4)}{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19}.$$

Alternate solution: Each sample point will represent the pair of the card that Eve turns up as well as the hand that Frank gets (i.e., the set of cards he receives regardless of the order he receives them). For instance, one sample point in $A \cap F$ is $(A\spadesuit, \{9\clubsuit, 10\clubsuit, 9\diamondsuit, 10\heartsuit, 10\heartsuit\})$. Calculate:

$$\Pr[A \cap F] = \frac{|A \cap F|}{|\Omega|} = \frac{4 \cdot \binom{8}{5}}{24 \cdot \binom{23}{5}} = \frac{1}{6} \cdot \frac{\binom{8}{5}}{\binom{23}{5}}.$$

(e) What is the (conditional) probability that Frank was dealt a hand with no face cards, given that the card Eve turned up was an ace?

$$\Pr[A|F] = \frac{\Pr[A \cap F]}{\Pr[A]} = \frac{\binom{8}{5}}{\binom{23}{5}}.$$