Solutions to the first midterm

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Problem 1 (30 points). Mark the following as true or false. To get full credit you must justify your answer. Each question is worth 6 points. A correct answer with no justification is worth 2 points, and a good justification may earn you points even if your answer is wrong.

(a) Every finitely generated abelian group of order 8 is isomorphic to either $\mathbb{Z}/8\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Solution: False; $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is another possibility.

(b) Every subgroup of S_3 is abelian.

Solution: False; S_3 is not itself abelian. It is true, however, that every *proper* subgroup of S_3 is abelian.

(c) Every permutation in S_n can be factored into a product of disjoint transpositions.

Solution: False; A transposition has order 2, and since disjoint cycles commute, any product of disjoint transpositions must also have order 2. In particular, a permutation of order 3 can not be factored into a product of disjoint transpositions.

(d) The symmetry group of a regular heptagon (7-gon) has a subgroup of order 7.

Solution: True; The subgroup generated by a rotation by $2\pi/7$ has order 7.

(e) $\mathbb{C} \setminus \{0\}$ has a non-trivial finite subgroup.

Solution: True; As an example, $\{-1, 1, -i, i\}$ is a subgroup of order 4. In fact, $\mathbb{C} \setminus \{0\}$ has a subgroup of order n for each natural number n.

Problem 2 (25 points). Let $A = \{a, b, c, d\}$ be a set with 4 elements, and let $\sigma \in S_A$ be a permutation satisfying that $\sigma(a) = a$.

(a) **(15 points).** Show that σ can not have order 4.

Solution: Since σ fixes a, we may regard σ as a permutation in $S_{\{b,c,d\}}$, which is a group of order 6. Since the order of an element divides the order of the group (Theorem 10.12), it follows that σ can not have order 4.

(b) **(10 points).** For each $n \in \{1, 2, 3\}$, give a concrete example where σ has order n.

Solution: The identity has order 1, (b, c) has order 2, and (b, c, d) has order 3. All these permutations fix a.

Problem 3 (20 points). Determine the order of the following groups:

(a) **(5 points).** $S_4 \times \mathbb{Z}/7\mathbb{Z}$.

Solution: S_4 has order $4! = 24$ and $\mathbb{Z}/7\mathbb{Z}$ has order 7. Hence, $S_4 \times \mathbb{Z}/7\mathbb{Z}$ has order 168.

(b) **(5 points).** The symmetry group of the plus symbol, i.e. the subset of $\mathbb{R}^2$ given by $\{(s,t) \in \mathbb{R}^2 \mid st = 0,\ -1 \le s \le 1,\ -1 \le t \le 1\}$.

Solution: There are 4 reflections (horizontal, vertical, and two diagonal) and 4 rotations, so the symmetry group has order 8. It is isomorphic to the symmetry group of the square.

(c) **(5 points).** The subgroup of $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$ generated by $(3, 8, 10)$.

Solution: Using Theorem 6.14, we see that 3 has order $4/\gcd(4,3) = 4$ in $\mathbb{Z}/4\mathbb{Z}$, 8 has order $10/\gcd(10,8) = 5$ in $\mathbb{Z}/10\mathbb{Z}$, and 10 has order $12/\gcd(12,10) = 6$ in $\mathbb{Z}/12\mathbb{Z}$. By Theorem 11.9, the order of $(3, 8, 10)$ in $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$ is $\operatorname{lcm}(4, 5, 6) = 60$.

(d) **(5 points).** The factor group G/H, where $G = \mathbb{Z}/9\mathbb{Z}$ and H is the subgroup of G generated by 6.

Solution: Since 6 has order 3 in $\mathbb{Z}/9\mathbb{Z}$, H has order 3. It follows that the order of G/H is $9/3 = 3$.

Problem 4 (25 points). Let $\mathrm{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ matrices. Let $\mathrm{SL}_n(\mathbb{R})$ be the subset of $\mathrm{GL}_n(\mathbb{R})$ consisting of matrices with determinant 1.

(a) **(10 points).** Show that $\mathrm{SL}_n(\mathbb{R})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$.

Solution: By definition, $\mathrm{SL}_n(\mathbb{R})$ is the kernel of the homomorphism

$$\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R} \setminus \{0\}.$$

By Corollary 13.20, $\mathrm{SL}_n(\mathbb{R})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$.

(b) **(15 points).** Show that the factor group $\mathrm{GL}_n(\mathbb{R}) / \mathrm{SL}_n(\mathbb{R})$ is isomorphic to $\mathbb{R} \setminus \{0\}$.

Solution: By Theorem 14.11, $\mathrm{GL}_n(\mathbb{R}) / \mathrm{SL}_n(\mathbb{R})$ is isomorphic to the image of the determinant homomorphism. Since this homomorphism is surjective (for any $a \in \mathbb{R} \setminus \{0\}$, we have $a = \det \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})$, the result follows.