

1 Given:  $M, \rho, c, \epsilon, \sigma, T(t=0) = T_i$

a) • heat lost through radiation

$$\frac{dQ}{dt} = -\epsilon \sigma A T^4$$

( $T$  - (negative sign - to account for) heat lost)

• Note  $\frac{dQ}{dt} \neq$  constant since  $T$  changes

• taking the differential form of specific heat equation ( $Q = mc \Delta T$ )

$$dQ = M c dT$$

• plug  $dQ$  into radiation equation

$$M c \frac{dT}{dt} = -\epsilon \sigma A T^4$$

• separation of variables and integrate from initial condition to final condition

$$\frac{-M c}{\epsilon \sigma A} \int_{T_i}^{T_0} \frac{1}{T^4} dT = \int_0^t dt$$

• must solve for  $A$  in terms of initial quantities.

$$\rho = \frac{M}{V}$$

$$V = \frac{4}{3} \pi r^3$$

$$A = 4\pi r^2$$

$$\frac{M c}{3 \epsilon \sigma A} \left. \frac{1}{T^3} \right|_{T_i}^{T_0} = t$$

$$\frac{1}{T^3} \Big|_{T_i}^{T_0} = t$$

$$\rho = \frac{3M}{4\pi r^3}$$

$$r = \left( \frac{3M}{4\pi \rho} \right)^{1/3}$$

$$\frac{M c}{3 \epsilon \sigma A} \left( \frac{1}{T_0^3} - \frac{1}{T_i^3} \right) = t$$

$$A = 4\pi \left( \frac{3M}{4\pi \rho} \right)^{2/3}$$

$$t = \frac{M c}{3 \epsilon \sigma 4\pi \left( \frac{3M}{4\pi \rho} \right)^{2/3}} \left( \frac{1}{T_0^3} - \frac{1}{T_i^3} \right)$$

plug in values, get

$$t = 2.5 \times 10^{12} \text{ sec}$$

b) Net heat loss calculated (through calorimetry equation)

$$Q = mc \Delta T \quad \text{where } T \text{ goes from } T_1 \text{ to } T$$

$$|Q_{\text{net loss}}| = Mc(T_1 - T)$$

c) Start with radiation equation:  $\frac{dQ}{dt} = \epsilon \sigma A T^4$

$$\text{heat released: } -\epsilon \sigma A T^4$$

$$\text{heat absorbed: } \epsilon \sigma A T_0^4$$

net rate

$$\frac{dQ}{dt} = \epsilon \sigma A (T_0^4 - T^4)$$

differential form of calorimetry equation:  $dQ = mc dT$

$$mc \frac{dT}{dt} = \epsilon \sigma A (T_0^4 - T^4)$$

plug in value of  $A$ :  $A = 4\pi \left(\frac{3M}{4\pi\rho}\right)^{2/3}$

$$\frac{dT}{dt} = \frac{\epsilon \sigma 4\pi \left(\frac{3M}{4\pi\rho}\right)^{2/3}}{mc} (T_0^4 - T^4)$$

part a) 13 pts

b) 6 pts

c) 6 pts

25 pts total

## Problem 2 Solution

2a) The temperature and pressure are given, so we can find the number of molecules per unit volume directly from the ideal gas law,

$$\frac{N}{V} = \frac{P}{kT} = \frac{10^5 \text{ Pa}}{1.38 \times 10^{-23} \text{ J/K} \times 273 \text{ K}} = \boxed{2.65 \times 10^{25} \frac{1}{\text{m}^3}}$$

2b) Recall the definition of root-mean-squared speed is,  $v_{\text{rms}} = \sqrt{\langle v^2 \rangle}$ . To calculate the average speed squared we use the Maxwell-Boltzmann distribution,

$$\langle v^2 \rangle = \frac{1}{N} \int_0^\infty 4N\pi \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}} v^2 \times v^2 dv \quad (1)$$

$$= 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty e^{-\frac{mv^2}{2kT}} v^4 dv \quad (2)$$

This is an example of a gaussian integral. Gaussian integrals will be the most common form of integral you will encounter in physics and for this reason it is well worth your time to memorize the fundamental result:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

As always you can look up this and other gaussian integrals but there is a useful technique for generating all gaussian integrals from this one. The technique is to treat  $a$  as a parameter and take the derivative with respect to  $a$ . To get the integral of interest in this problem we have to do this twice:

$$\begin{aligned} \frac{d}{da} \left( \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \right) \\ \Rightarrow \int_{-\infty}^{\infty} -x^2 e^{-ax^2} dx = -\frac{\sqrt{\pi}}{2} a^{-3/2}, \end{aligned}$$

Because the function we're integrating is the same from  $-\infty$  to 0 as it is from 0 to  $\infty$  (an even function) we can multiply it by 2 and integrate from 0 to  $\infty$ :

$$\begin{aligned} 2 \int_0^\infty x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{2} a^{-3/2} \\ \Rightarrow \int_0^\infty x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4} a^{-3/2}. \end{aligned}$$

Finally, using the derivative technique again we obtain,

$$\begin{aligned} \frac{d}{da} \left( \int_0^\infty x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4} a^{-3/2} \right) \\ \Rightarrow \int_0^\infty x^4 e^{-ax^2} dx = \frac{3\sqrt{\pi}}{8} a^{-5/2}, \end{aligned}$$

which is the integral we were looking to find. Plugging this result into Eq. (2) gives,

$$\langle v^2 \rangle = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} \frac{3\sqrt{\pi}}{8} \left( \frac{m}{2kT} \right)^{-5/2} = \frac{3}{2} \frac{2kT}{m} = \frac{3kT}{m}.$$

Then the root-mean-squared speed is,

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}}.$$

It's nice to see the derivation of this result from scratch but this was not required to get full credit on this problem.

2c) See section 18-6 of your text for a complete discussion of this problem including illustrations. Consider a cylinder of radius  $R_c = R + r$ , any nitrogen molecules whose centers are within this cylinder will be struck by a menthol molecule moving through it. The length of the cylinder is given by the average speed of the menthol molecules times the time interval under consideration, say  $\Delta t$ , *i.e.*  $L_c = \bar{v}\Delta t$ . The volume of this cylinder is  $V_c = \pi R_c^2 L_c = \pi(R+r)^2 \bar{v}\Delta t$ . The number of collisions that a menthol molecule will experience, call it  $s$ , is just the number of nitrogen molecules in this cylinder, given that the density of the nitrogen is  $N/V$  this is,  $s \equiv \# \text{collisions} = N/V \cdot \pi(R+r)^2 \bar{v}\Delta t$ . Then the mean free path is,

$$l = \frac{\bar{v}\Delta t}{N/V \cdot \pi(R+r)^2 \bar{v}\Delta t} = \frac{1}{\pi(R+r)^2 \left(\frac{N}{V}\right)}.$$

The factor of  $\sqrt{2}$  mentioned in the statement of the problem comes from taking into account the fact that the nitrogen molecules are moving and not stationary. This reasoning was not required to get full credit.

2d) Given that the displacement  $x$ , mean free path  $l$  and number of collisions  $s$  are related by  $x^2 = sl^2$  we can substitute in what we know about number of collision and mean free path in a time  $t$  and find,

$$x^2 = \frac{N}{V} \pi(R+r)^2 \bar{v}t \cdot \frac{1}{(\pi(R+r)^2 \frac{N}{V})^2} = \frac{\bar{v}t}{\pi(R+r)^2 \frac{N}{V}}.$$

Now, since we are given displacement  $x$  we can invert this to find the time  $t$ ,

$$t = \frac{\pi(R+r)^2 \frac{N}{V} x^2}{\bar{v}},$$

and because we know  $\bar{v} = \sqrt{8kT/M\pi}$  we have everything that we need to find the time.

2e) Plugging in the various numbers yields,

$$t = \frac{\pi(R+r)^2 \frac{N}{V} x^2}{\bar{v}} = \frac{2.13 \times 10^{10} \text{m}}{192 \text{m/s}} = \boxed{1.1 \times 10^8 \text{s}}.$$

A neat and useful computational fact is that there are  $\pi \times 10^7 \text{sec/year}$ , this is accurate to 1%. Putting this in we get,

$$t = \boxed{3.5 \text{years}}.$$

2f) This is an extremely long time and I suspect that it does not agree with your experience that you can smell the menthol only briefly after someone across the room opens the jar. The reason for this is that collisional diffusion is not the predominant effect for spreading the menthol molecules around the room. Instead there are usually significant convection currents (breezes on the scale of the room) that carry the molecules to your nose more quickly. You can read more about convection in section 19-10.

### Problem 3

**Part A:**

In this configuration, the heat output of the first engine is the heat input of the second engine. Let  $e_1 = \frac{1}{2}(1 - \frac{T_2}{T_1})$  and  $e_2 = \frac{1}{2}(1 - \frac{T_4}{T_3})$  be the efficiencies of the first and second engines respectively. Note that each engine is operating at 50% of its Carnot efficiency, as stated in the problem.

$$\begin{aligned} Q_{H1} &= W_1 + Q_{L1} \\ W_1 &= e_1 \cdot Q_{H1} \Rightarrow Q_{L1} = (1 - e_1) \cdot Q_{H1} \\ W_2 &= e_2 \cdot Q_{H2} = e_2 \cdot Q_{L1} = e_2 \cdot (1 - e_1) \cdot Q_{H1} \\ W_{total} &= W_1 + W_2 = (e_1 + e_2 - e_1 \cdot e_2) Q_{H1} \end{aligned}$$

Let  $r$  be the rate at which coal is being burned. Then  $\frac{\Delta Q_{H1}}{\Delta t} = rb$ .

$$\begin{aligned} p &= \frac{\Delta W_{total}}{\Delta t} = (e_1 + e_2 - e_1 \cdot e_2) \cdot \frac{\Delta Q_{H1}}{\Delta t} \\ \Rightarrow p &= (e_1 + e_2 - e_1 \cdot e_2) \cdot rb \\ \Rightarrow r &= \frac{p}{(e_1 + e_2 - e_1 \cdot e_2) \cdot b} = 122 \frac{kg}{s} \end{aligned}$$

■

**Part B:**

For this part, we'll look at the engine as a whole and apply the first law of thermodynamics.

$$\begin{aligned} Q_{in} &= W + Q_{out} \Rightarrow \frac{\Delta Q_{in}}{\Delta t} = \frac{\Delta W}{\Delta t} + \frac{\Delta Q_{out}}{\Delta t} \\ \Rightarrow rb &= p + \frac{\Delta Q_{out}}{\Delta t} = p + \frac{\Delta Q_W}{\Delta t} \\ \Rightarrow \frac{\Delta Q_W}{\Delta t} &= rb - p = 2665 \text{ MW} \end{aligned}$$

■

**Part C:**

$$Q = mc\Delta T \Rightarrow \frac{\Delta Q_W}{\Delta t} = \frac{\Delta m}{\Delta t} c_W \Delta T$$
$$\Rightarrow \frac{\Delta m}{\Delta t} = \frac{\frac{\Delta Q_W}{\Delta t}}{m_W \Delta T} = 106000 \frac{\text{kg}}{\text{s}}$$

■

## 1 Huang, Spring 2009 Problem 4 (25 points)

An ideal monoatomic gas in a thermally insulated box is separated by a thermally conducting partition into two parts. There are  $n$  moles of gas in each part. Initially the gas in part A has temperature  $T_1$  and volume  $V_1$  and in B temperature  $T_2$  and volume  $V_2$ . The partition can slide without friction and the two parts have the same pressure,  $P$ .

(a) (10 pts.) What is the final temperature when thermal equilibrium is reached?

**Solution 1:** The partition is free to slide and hence will slide in order to balance the forces it feels from the gas particles in each of the two partitions. Therefore,

(1 pt.) This is a constant pressure (isobaric) process.

(1 pt.) At thermal equilibrium, the temperatures of the two partitions,  $T$ , are the same.

(1 pt.) By the Ideal Gas Law, the final volumes of the two partitions are the same,  $\frac{V_1+V_2}{2}$ .

(5 pts.) IGL:  $PV_1 = nRT_1$  and  $PV_2 = nRT_2$  and  $P\frac{V_1+V_2}{2} = nRT$ .

(2 pts.) Combining these gives

$$T = \frac{PV_1}{2nR} + \frac{PV_2}{2nR} = \frac{T_1}{2} + \frac{T_2}{2} = \frac{T_1 + T_2}{2}$$

**Solution 2:** We can use the heat capacity at constant pressure

(1 pt.) At thermal equilibrium, the temperatures of the two partitions,  $T$ , are the same.

(1 pt.) This is a constant pressure (isobaric) process. Therefore, to calculate heat, we have to use the heat capacity at constant pressure,  $C_P$ .

(1 pt.) The gases on either side do not have to be the same, they just have to be ideal. For ideal monoatomic gases,  $C_P = \frac{5}{2}nR$ .

(5 pts.) Since the box is thermally insulated, no heat enters or leaves the WHOLE system (including both gas A and B). Therefore, the any heat absorbed by one must be lost by the other:  $C_P\Delta T_A = \frac{5}{2}nR(T - T_1) = Q_A = -Q_B = -\frac{5}{2}nR(T - T_2) = -C_P\Delta T_B$ .

(2 pts.) Solving for  $T$  gives

$$T - T_1 = -(T - T_2) \quad \implies \quad T = \frac{T_1 + T_2}{2}$$

**Solution 3:** This is almost exactly like Solution 2, but instead of setting heats equal, we set  $\Delta E_{\text{int}}$  for the entire system equal to 0. This must be justified once again by the fact that the box is thermally insulated AND does not expand or contract as a whole (and so there is no net work for the system).

**(b) (12 pts.)** Calculate the entropy change for the gas in A and for the gas in B. (Please specify the reversible path that you choose for this calculation.)

**Solution 1:** The simplest reversible path to choose to simply describe the system is:

(3 pts.) Isobaric compression (for the one that started out hotter) or expansion (for the one that started out colder)

However, if one were to actually calculate the entropy change from the reversible path, one simpler choice would be isothermal expansion followed by isochoric heating (for the one that started out colder) and isothermal compression followed by isochoric cooling (for the one that started out hotter):

(6 pts.) Suppose  $T_1 < T_2$  and so  $V_1 < V_2$ . For the initially colder A gas, the isothermal expansion connects the two points  $(V_1, P, T_1)$  and  $(\frac{V_1+V_2}{2}, \frac{2V_1}{V_1+V_2}P, T_1)$ . We found the pressure coordinate of the second point by setting  $PV = \text{const.}$  along an isotherm. Along this isotherm, the internal energy doesn't change and so  $Q_{\text{isotherm}} = W_{\text{isotherm}} = nRT_1 \ln \frac{V_f}{V_i} = nRT_1 \ln \frac{V_1+V_2}{2V_1}$ . Since temperature doesn't change along this curve, the change in entropy here is just  $\Delta S_{\text{isotherm}} = \frac{Q_{\text{isotherm}}}{T_1} = nR \ln \frac{V_1+V_2}{2V_1}$ . Note that from the two IGL equations  $PV_i = nRT_i$ , we conclude that  $T_1/T_2 = V_1/V_2$  and so we can write  $\Delta S_{\text{isotherm}} = nR \ln \frac{T_1+T_2}{2T_1}$ .

Next, we have isochoric heating connecting  $(\frac{V_1+V_2}{2}, \frac{2V_1}{V_1+V_2}P, T_1)$  to  $(\frac{V_1+V_2}{2}, P, \frac{T_1+T_2}{2})$ . We have  $dQ_{\text{isochoric}} = d\Delta E_{\text{isochoric}} = \frac{3}{2}nRdT$  since no work is done in isochoric processes. Since temperature is no longer constant for this step, we need to work with differentials. We have  $\Delta S_{\text{isochoric}} = \int \frac{dQ_{\text{isochoric}}}{T} = \frac{3}{2}nR \int \frac{dT}{T} = \frac{3}{2}nR \ln \frac{T_f}{T_i} = \frac{3}{2}nR \ln \frac{T_1+T_2}{2T_1}$ .

(2 pts.) Finally, the total entropy change is

$$\Delta S_A = \Delta S_{\text{isothermal}} + \Delta S_{\text{isochoric}} = \frac{5}{2}nR \ln \frac{T_1 + T_2}{2T_1}$$

Completely analogously, we find,

$$\Delta S_B = \frac{5}{2}nR \ln \frac{T_1 + T_2}{2T_2}$$

You could also just use the formula for the entropy change for an ideal gas which works whether or not the process is reversible:

$$\Delta S = \frac{d}{2}nR \ln \frac{T_f}{T_i} + nR \ln \frac{V_f}{V_i}$$

(1 pt.) Realizing that  $d = 3$  for monoatomic ideal gas.



**Solution 2 for part (b):** Again, a maximum of 3 points are awarded for the specification of a reversible path. Then, we realize that we can use the equation  $\Delta S = \int \frac{dQ}{T} = C_P \int \frac{dT}{T} = \frac{5}{2}nR \ln(T_f/T_i)$  for a reversible isobaric process. This reproduces the previous results. It must be clear that you are using  $C_P$  and not  $C_V$  or any other heat capacity and that  $C_P = \frac{5}{2}nR$  is the same for all ideal monoatomic gases.

(c) (3 pts.) Calculate the entropy change of the combined system of A and B,  $\Delta S_{\text{total}}$ .

**Solution:** You need to add the results of part (b):

(1 pt.)  $\Delta S_{\text{total}} = \Delta S_A + \Delta S_B$

(2 pts.)  $\Delta S_{\text{total}} = \frac{5}{2}nR \ln \frac{(T_1+T_2)^2}{4T_1T_2}$

(d) (4 bonus pts.) Show that  $\Delta S_{\text{total}} \geq 0$ .

**Solution 1:** Since it is a square, we know  $(T_1 - T_2)^2 \geq 0$ . Expanding the left hand side gives  $T_1^2 - 2T_1T_2 + T_2^2 \geq 0$ . Adding  $4T_1T_2$  to both sides gives  $T_1^2 + 2T_1T_2 + T_2^2 \geq 4T_1T_2$  or  $(T_1 + T_2)^2 \geq 4T_1T_2$ . Assuming we are not in a crazy situation where exactly one of the  $T_i$ 's is negative (on the absolute Kelvin scale; what does that mean?), then we have  $\frac{(T_1+T_2)^2}{4T_1T_2} \geq 1$  which implies  $\ln \frac{(T_1+T_2)^2}{4T_1T_2} \geq 0$  and so  $\Delta S_{\text{total}} \geq 0$ .

**Solution 2:** Again, we want to prove  $\frac{(T_1+T_2)^2}{4T_1T_2} \geq 1$ . Multiply by 4, expand the square and cancel terms from the numerator and denominator to get  $\frac{T_1}{T_2} + 2 + \frac{T_2}{T_1} \geq 4$  or  $x + \frac{1}{x} \geq 2$  where  $x = T_1/T_2$ . We want to prove this inequality for all positive  $x$ . As  $x \rightarrow \infty$  and  $x \rightarrow 0$ , we have  $x + \frac{1}{x} \rightarrow \infty$ . Therefore, there must be at least one local minimum somewhere and if there is exactly one then it has to be a global minimum.

Well, setting the derivative equal to zero gives  $1 - \frac{1}{x^2} = 0$  and  $x = 1$  is the only positive solution. We automatically know that it is a minimum by the previous argument, but we can check the second derivative:  $\frac{2}{x^3} \Big|_{x=1} = 2 > 0$  which shows that  $x = 1$  is a local minimum. Since there is only one,  $x = 1$  is a global minimum. But,  $1 + \frac{1}{1} = 2 \geq 2$ . Thus, we have proven the inequality.