Midterm

1. (25 points) Water for a city is supplied from two sources; namely, Source A and Source B. During the summer season, the probability that the supply from Source A will be below normal is 0.30; the corresponding probability for Source B is 0.15. However, if Source A is below normal, the probability that Source B will also be below normal during the same summer season is increased to 0.30.

   The probability of water shortage in the city will obviously depend on the supplies from the two sources. In particular, if only Source A is below normal supply, the probability of water shortage is 0.20, whereas if only Source B is below normal the corresponding probability of shortage is 0.25. Obviously, if none of the sources are below normal, there would be no chance of shortage, whereas if both sources are below normal during the summer, the probability of water shortage in the city would be 0.80.

   What is the probability of water shortage in the city during the summer season?

i) Let's denote event A and B as follows

A: the event that water source A is below normal

B: the event that water source B is below normal **(2points)**

Then,

$P(A) = 0.3$, $P(B) = 0.15$

$P(B|A) = 0.3$ **(3points)**

ii) Let's denote event S as follows

S: the event that water shortage occurs in the city **(1points)**

Then,

$P(S|A\bar{B}) = 0.2, P(S|\bar{A}B) = 0.25, P(S|\bar{A}\bar{B})=0, P(S|AB)=0.8$ **(4points)**

iii) $P(S) = P(S|A\bar{B})P(A\bar{B}) + P(S|\bar{A}B)P(\bar{A}B) + P(S|\bar{A}\bar{B})P(\bar{A}\bar{B}) + P(S|AB)P(AB)$ **(5points)**

Where,

$P(AB) = P(B|A)P(A) = (0.3)(0.3) = 0.09$ **(3points)**

$P(A\bar{B}) = P(A) - P(AB) = 0.3 - 0.09 = 0.21$ **(2points)**

$P(\bar{A}B) = P(B) - P(AB) = 0.15 - 0.09 = 0.06$ **(2points)**

$\therefore P(S) = (0.2)(0.21) + (0.25)(0.06) + 0 + (0.8)(0.09) = \underline{0.129}$ **(3points)**

2. (15 points) The probability that a strong earthquake will cause damage to a certain structure has been estimated to be 0.02. What is the probability that the structure will be damaged during the second of two such strong earthquakes? It may be assumed that damages to the structure caused by successive earthquakes are statistically independent. (Specify any additional assumption that you make.)

[Solution 1]

i) Let's denote event D1 and D2 as follows

D1: the event that the structure will be damaged by the first strong earthquake

D2: the event that the structure will be damaged by the second strong earthquake **(3points)**

ii) If we assume that the structure is not reconstructed after it is damaged by a strong earthquake, **(3points)**

iii) the probability that the structure will be damaged during the second of two such strong earthquakes:

$P(\overline{D1}D2) = P(\overline{D1})P(D2)$=(0.98)(0.02)=<u>0.0196</u> (because of independence assumption) **(9points)**

[Solution 2]

i) Let's denote event D1 and D2 as follows

D1: the event that the structure will be damaged by the first strong earthquake

D2: the event that the structure will be damaged by the second strong earthquake **(3points)**

ii) If we assume that the structure is either undamaged or reconstructed before the second earthquake, **(3points)**

iii) the probability that the structure will be damaged during the second of two such strong earthquakes:

$P(D2) = P(\overline{D1}D2) + P(D1D2)$=<u>0.02</u> **(9points)**

3. (30 points) Severe snow storm is defined as a storm whose snowfall exceeds 10 inches. Let X be the amount of snowfall in a severe snow storm. The cumulative distribution function (CDF) of X in a given town is

$$F_X(x) = 1 - (\frac{10}{x})^4 \qquad \text{for } x \geq 10$$
$$= 0\,; \qquad \text{for } x < 10$$

(1) Determine the median of X

Let's denote the median of X as x_m.
Then, x_m can be obtained by using CDF as below.

$P(X \leq x_m) = 0.5$ **(5points)**

$\leftrightarrow F_X(x_m) = 0.5$

$\leftrightarrow 1 - (\frac{10}{x_m})^4 = 0.5$ **(8points)**

$\leftrightarrow x_m = \underline{11.9}\ (inches)$ **(2points)**

(2) Suppose a *disastrous snow storm* is defined as a storm with over 15 inches of snowfall. What percentage of the severe snow storms are disastrous?

Let's denote event A as follows.

A: the event that a disastrous snow storm occurs

Then, the probability that the severe snow storms are disastrous is

$P(A) = P(X > 15)$ **(5points)**

$= 1 - P(X \leq 15)$

$= 1 - \left(1 - \left(\frac{10}{15}\right)^4\right)$ **(8points)**

$= (\frac{2}{3})^4 = \underline{0.198}$ **(2points)**

4. (30 points) The truck traffic on a certain highway can be described as a Poisson process with a mean arrival rate of 1 truck per minute. The weight of each truck is random, and the probability that a truck is overloaded is 10%.

(1) What is the probability that there will be at least two trucks passing a weigh station on this highway in a 5-min period?

It is known that truck arrival has mean rate $v = 1\ (truck/minute)$
This means that average number of trucks' arrival is 5 per every 5 minute

i) Let X be the number of trucks which arrive on the freeway in a 5-minute period. **(1points)**

ii) Hence,

$P(X \geq 2) = 1 - P(X < 2)$ **(5points)**

$= 1 - P(X = 0) - P(X = 1)$ **(3points)** (can be omitted)

$= 1 - e^{-5} - 5e^{-5}$ **(4points)**

$\approx$ 0.96 **(2points)**

(2) What is the probability that at most one of the next five trucks stopping at the weigh station will be overloaded?

It is known that the number of trucks (n) is 5; and the probability that a truck is overloaded (p) is 0.1

i) Let K be the number of trucks that will be overloaded among the next n trucks. **(1points)**

Then, we can assume that K follows binomial distribution **(2points)**

i.e. K~B(n,p)=B(5,0.1)

ii) Hence,

$P(K \leq 1)$ **(3points)**

$= P(K = 0) + P(K = 1)$ **(3points)** (can be omitted)

$= (1 - 0.1)^5 + 5 \cdot (0.1)^1 (1 - 0.1)^4$ **(4points)**

= 0 .92 **(2points)**