SÜSS MicroTec

Prob. 1 $\Delta P = RQ$

Hagen-Poiseuille

$$Q = \frac{\pi}{128\mu}\frac{\Delta P}{\Delta x}D^4$$

a) $$\underline{\frac{\Delta P}{Q} = R = \frac{128\mu}{\pi}\frac{L}{D^4}}$$ 5

Tube 1. $$R_1 = \frac{128}{\pi}\mu\left(\frac{L_1}{D_1^4} + \frac{L_2}{D_2^4}\right)$$

Tube 2. $$R_2 = \frac{128}{\pi}\mu\frac{(L_1+L_2)}{D_1^4}$$

a. know ΔP find Q

$$Q_1 = \frac{\Delta P}{R_1} = \frac{\Delta P\cdot\pi}{128\mu\left(\frac{L_1}{D_1^4} + \frac{L_2}{D_2^4}\right)}$$

$$= \frac{\Delta P\cdot\pi\cdot D_1^4 D_2^4}{128\mu\left(L_1 D_2^4 + L_2 D_1^4\right)}$$ 5

$$Q_2 = \frac{\Delta P}{R_2} = \frac{\Delta P\cdot\pi\cdot D_1^4}{128\mu(L_1+L_2)}$$

$Q_2 > Q_1 \Rightarrow \frac{Q_1}{Q_2} < 1$

$$\frac{Q_1}{Q_2} = \frac{\cancel{\Delta P}\cdot\cancel{\pi}}{\cancel{128\mu}}\frac{D_1^4 D_2^4}{(L_1 D_2^4 + L_2 D_1^4)}\cdot\frac{\cancel{128\mu}(L_1+L_2)}{\cancel{\Delta P}\cdot\cancel{\pi}\cdot D_1^4}$$

$$= \frac{\cancel{D_1^4}D_2^4(L_1+L_2)}{\cancel{D_1^4}(L_1 D_2^4 + L_2 D_1^4)}$$

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(2)

Prob. 1

if $L_2 = 0 \Rightarrow \frac{Q_1}{Q_2} = 1$ makes sense

if $L_1 = 0 \Rightarrow \frac{Q_1}{Q_2} = \frac{D_2^4 L_2}{D_1^4 L_2} = \frac{D_2^4}{D_1^4}$

know $D_2 < D_1 \Rightarrow \frac{Q_1}{Q_2} \leq 1$ (10)

(5)

b. given ΔP which tube has a higher V_{max}

$V_{max} \approx \bar{V}$

$$\bar{V} = \frac{4Q}{\pi D_2^2}$$

$$\bar{V}_1 = \frac{4Q_1}{\pi D_2^2} = \frac{4}{\pi D_2^2} \frac{\Delta P \pi}{128 \mu} \frac{D_1^4 D_2^4}{(L_1 D_2^4 + L_2 D_1^4)}$$

$$= \frac{\Delta P}{32 \mu} \frac{D_1^4 D_2^2}{(L_1 D_2^4 + L_2 D_1^4)}$$ (5)

$$\bar{V}_2 = \frac{4Q_2}{\pi D_1^2} = \frac{\Delta P}{32 \mu} \frac{D_1^2}{(L_1 + L_2)}$$

as $L_2 \to 0$

$$\bar{V}_1 = \frac{\Delta P}{32 \mu} \frac{D_1^4 D_2^2}{L_1 D_2^4} = \frac{\Delta P}{32 \mu} \frac{1}{L_1} \frac{D_1^4}{D_2^2}.$$

$$\bar{V}_2 = \frac{\Delta P}{32 \mu} \frac{D_1^2}{L_1}$$

$$\frac{\bar{V}_1}{\bar{V}_2} = \frac{D_1^4}{D_2^2} \frac{1}{D_1^2} = \frac{D_1^2}{D_2^2} > 1$$ (5)

$D_2 < D_1$

if $L_1 \to 0$

$$\bar{V}_1 = \frac{\Delta P}{32\mu}\,\frac{D_1^4 D_2^2}{L_2 D_1^4} = \frac{\Delta P}{32\mu}\,\frac{1}{L_2}\,D_2^2$$

$$\bar{V}_2 = \frac{\Delta P}{32\mu}\,\frac{D_1^2}{L_2}$$

$$\frac{\bar{V}_1}{\bar{V}_2} = \frac{D_2^2}{D_1^2} \qquad D_2 < D_1$$

$$\frac{\bar{V}_1}{\bar{V}_2} < 1$$

At some L_1 & L_2 $\bar{V}_1 = \bar{V}_2$

$$1 = \frac{D_1^4 D_2^2}{L_1 D_2^4 + L_2 D_1^4}\cdot\frac{(L_1 + L_2)}{D_1^2}$$

$$= \frac{D_1^2 D_2^2 \,(L_1 + L_2)}{L_1 D_2^4 + L_2 D_1^4}$$

There are multiple solns for various D_1 & D_2

5

c. if Q is fixed

$\Rightarrow Q_1 = Q_2$ so they both have the same flow rate

5

d. Tube 1 has a higher V_{max} by a factor $\frac{D_1^2}{D_2^2}$

cons. of mass

5

pressure $\Rightarrow$ $Q_2 > Q_1$

$V_{max,1}$ & V_{max_2} are not unique

Hi Dorian!

So the shortest cut for problem 1 b) a. is knowing that $\overline{u_z} = \frac{1}{2} \cdot u_{max}$. This can be derived as follows:

The reduces NV-equation for the round pipe case on polar cylinder coordinates is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z}$$

The solution (integrate twice) of which is given by: $u_z = \frac{1}{4\mu}\frac{\partial p}{\partial z} r^2 + c_1 \ln r + c_2$

Since at $r = 0$ (middle of the tube) the velocity cannot diverge $c_1 = 0$. The no-slip boundary conditions at $r = R$ (pipe wall) lead to $c_2 = -\frac{1}{4\mu}\frac{\partial p}{\partial z} R^2$.

$$\Rightarrow \quad u_z = -\frac{1}{4\mu}\frac{\partial p}{\partial z}(R^2 - r^2)$$

The velocity is obviously highest in the middle at $r = 0$: $\quad u_{z\max} = -\frac{1}{4\mu}\frac{\partial p}{\partial z} R^2$

The average velocity can be simply obtained by integration over the pipe radius and dividing by the cross-sectional area of the pipe:

$$\overline{u_z} = \frac{1}{\pi R^2}\int_0^R u_z \cdot 2\pi r dr = \frac{1}{\pi R^2}\int_0^R \left[-\frac{1}{4\mu}\frac{\partial p}{\partial z}(R^2 - r^2)\right] \cdot 2\pi r dr = \frac{1}{2} \cdot \left[-\frac{1}{4\mu}\frac{\partial p}{\partial z} R^2\right] = \frac{u_{z\max}}{2}.$$

Many students remembered actually from their basic fluid mechanics course that $\overline{u_z} = \frac{1}{2} \cdot u_{z\max}$ ... This makes the problem really simple since $\overline{u_z} = \frac{Q}{\pi R^2}$ where Q is given by $Q = \frac{\Delta P}{\mathcal{R}}$ from Ohm's law of fluid mechanics (an expression $\mathcal{R}$ was to be derived in 1a) from Hagen-Poiseuille equation).

Combine all this and you'll get: $u_{z\max} = 2 \cdot \frac{1}{\mathcal{R}} \cdot \frac{1}{\pi R^2} \cdot \Delta P$

Thus, deciding in which tube the maximum exit velocity is higher comes down to evaluating for which tube the product of the above expression is larger as a function of parameters. Please note.

- For the simple tube an expression for $\mathcal{R}$ is just the result of 1a).
- One has to treat the composite tube simply as two resistors n series by analogy to electronics.
- When evaluating $u_{z\,\mathrm{exit}}$ using the above expression for the simple tube ΔP is just the "given pressure drop". For the composite tube one has to use the pressure drop across the second segment *only*. This pressure drop can be calculated from the pressure drop across the entire system by using the "voltage divider rule" by analogy to electronics: $\Delta P_2 = \dfrac{\mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2}$.

problem 3

params

u (m/s)

h (m)

l (m)

$\mathcal{D}$ (m^2/s)

2 units m & s

$\Rightarrow$ 2 Π groups 5

$$\Pi_1 = \frac{u\,(m/s)}{\mathcal{D}\,(m^2/s)}\,h\,(m) = \frac{uh}{\mathcal{D}}$$ 5

$$\Pi_2 = \frac{h}{l}$$ 5

$$\frac{uh}{\mathcal{D}} = g\left(\frac{h}{l}\right)$$ 5

$$= f\left(\frac{l}{h}\right)$$

assume f is linear $\Rightarrow$ l increases U increases 5

$$\frac{uh}{\mathcal{D}} = c\,\frac{l}{h}$$ 5

$$\boxed{u = c\,\frac{l\,\mathcal{D}}{h^2}}$$