Prof. Adrian Lee Physics 7B Fall 07 Final Exam Solutions

Problem 1

a

The efficiency of the engine is a simple computation from the definition:

$$e_{\mathrm{r}} = \frac{W}{Q_{\mathrm{in}}} = \frac{600\ J}{1600\ J} = \frac{3}{8}\ . \tag{1}$$

("r" stands for "real" here.) The Carnot engine efficiency depends on the heat reservoir temperatures:

$$e_{\mathrm{C}} = \frac{T_{\mathrm{H}} - T_{\mathrm{L}}}{T_{\mathrm{H}}} = \frac{850\ K - 400\ K}{850\ K} = \frac{9}{17} \approx 0.53\ . \tag{2}$$

("C" stands for "Carnot" here.) Notice that the Carnot efficiency is bigger than the real efficiency.

b

The total entropy change of the universe consists of the entropy change of the engine and the entropy change of the environment. The engine itself is running through a cycle, so its entropy doesn't change because entropy is s state function. The environment, however, is not running through a cycle. Each of the heat reservoirs exchanges heat with the engine, but even though the reservoirs are losing/gaining heat they remain at the same temperature because they're so big. So we can use the fact that $\Delta S = Q/T$ when temperature is constant. The high temperature reservoir loses heat, and so loses entropy, while the low temperature reservoir gains heat (and entropy). In equations, we have

$$\begin{aligned} \Delta S_{\mathrm{universe}} &= \Delta S_{\mathrm{engine}} + \Delta S_{\mathrm{env}} = \Delta S_{\mathrm{env}} && (3) \\ &= \Delta S_{\mathrm{H}} + \Delta S_{\mathrm{L}} = -\frac{Q_{\mathrm{in}}}{T_{\mathrm{H}}} + \frac{Q_{\mathrm{out}}}{T_{\mathrm{L}}} && (4) \\ &= -\frac{Q_{\mathrm{in}}}{T_{\mathrm{H}}} + \frac{Q_{\mathrm{in}} - W}{T_{\mathrm{L}}} = -\frac{1600\ J}{850\ K} + \frac{1600\ J - 600\ J}{400\ K} && (5) \\ &\approx 0.62\ J/K\ . && (6) \end{aligned}$$

c

For a Carnot engine, we can use the same procedure as before except we will also use the relation $W = Q_{\rm in} e_{\rm C}$ which relates the work to the heat input. Note that the Q and W appearing here are not necessarily the same as those above.

$$\begin{aligned}
\Delta S_{\rm universe} &= \Delta S_{\rm engine} + \Delta S_{\rm env} = \Delta S_{\rm env} &(7)\\
&= \Delta S_{\rm H} + \Delta S_{\rm L} = -\frac{Q_{\rm in}}{T_{\rm H}} + \frac{Q_{\rm out}}{T_{\rm L}} &(8)\\
&= -\frac{Q_{\rm in}}{T_{\rm H}} + \frac{Q_{\rm in} - W}{T_{\rm L}} = -\frac{Q_{\rm in}}{T_{\rm H}} + \frac{Q_{\rm in} - Q_{\rm in} e_{\rm C}}{T_{\rm L}} &(9)\\
&= Q_{\rm in}\left(-\frac{1}{T_{\rm H}} + \frac{1 - e_{\rm C}}{T_{\rm L}}\right) = Q_{\rm in}\left(-\frac{1}{T_{\rm H}} + \frac{1}{T_{\rm L}} - \frac{1}{T_{\rm L}} + \frac{1}{T_{\rm H}}\right) &(10)\\
&= 0 \ . &(11)
\end{aligned}$$

d

The difference in work done, assuming they have the same energy input $Q_{\rm in}$, is equal to $Q_{\rm in}(e_{\rm C} - e_{\rm r})$. We can also go back to part b and use the formulas we got for the entropy change in the universe for a cycle of the real engine.

$$\begin{aligned}
\Delta S_{\rm universe} &= -\frac{Q_{\rm in}}{T_{\rm H}} + \frac{Q_{\rm in} - W}{T_{\rm L}} &(12)\\
&= -\frac{Q_{\rm in}}{T_{\rm H}} + \frac{Q_{\rm in} - Q_{\rm in} e_{\rm r}}{T_{\rm L}} &(13)\\
&= Q_{\rm in}\left(-\frac{1}{T_{\rm H}} + \frac{1 - e_{\rm r}}{T_{\rm L}}\right) \ . &(14)
\end{aligned}$$

Now we have

$$T_{\rm L}\Delta S_{\rm universe} = Q_{\rm in}\left(-\frac{T_{\rm L}}{T_{\rm H}} + 1 - e_{\rm r}\right) = Q_{\rm in}(e_{\rm C} - e_{\rm r}) = \Delta W \ . \qquad (15)$$

Problem 2

Throughout this problem we will normalize our potentials so that the negative side of the applied potential difference is at $V = 0$.

a

With S open, the capacitors will charge up and then there will be no current through them. So all of the current goes through the resistors. The total current is

$$I = \frac{V}{R_1 + R_2} \ . \qquad (16)$$

There is a voltage drop across R_2 between point a and the point of zero potential. So the potential at a must be equal in magnitude to the potential drop across R_2:

$$V_a = \frac{R_2}{R_1 + R_2} V \ . \tag{17}$$

b

By charge conservation, the charges on the two capacitors is the same; call it Q. Then the Loop Law through the capacitors says

$$V = \frac{Q}{C_1} + \frac{Q}{C_2} \ . \tag{18}$$

The potential at b must be equal in magnitude to the potential drop across C_2, so we get

$$V_b = \frac{Q}{C_2} = \frac{C_2^{-1}}{C_1^{-1} + C_2^{-1}} V \ . \tag{19}$$

c

After the switch is closed, charge can escape so we are no longer allowed to say that the capacitors carry equal charge. However, once they are fully charged to their new totals, there will be no current through them. So the resistors still carry all of the current, and we still have the loop law

$$I = \frac{V}{R_1 + R_2} \ . \tag{20}$$

Now the potential at b is the same as that at a, and is equal the potential drop across R_2:

$$V_b = \frac{R_2}{R_1 + R_2} V \ . \tag{21}$$

d

After the switch is thrown, we can no longer say that the charges on the two capacitors are equal. However, any difference in charge arises because some charge escaped by flowing across the switch. The total charge that flowed through the switch is then equal to the difference in the charges of the two capacitors after equilibrium is reached. Let the respective charges of the capacitors be Q_1 and Q_2. Since a and b are connected, the potential drop across R_1 has to equal that across C_1:

$$IR_1 = \frac{Q_1}{C_1} \ . \tag{22}$$

Similarly, the potential drop across R_2 has to equal that across C_2:

$$IR_2 = \frac{Q_2}{C_2} \ . \tag{23}$$

Now we can easily compute the charge difference:

$$Q_1 - Q_2 = IR_1C_1 - IR_2C_2 = \frac{R_1C_1 - R_2C_2}{R_1 + R_2} V \ . \tag{24}$$

Problem 3

The mutual inductance M can be computed in one of two ways. First, if we place a current I_1 in the larger solenoid and measure the magnetic flux Φ_2 in the smaller solenoid, we should have

$$\Phi_2 = MI_1 \ . \tag{25}$$

The other way we can compute it is to place a current I_2 in the smaller solenoid and measure the magnetic flux Φ_2 in the larger solenoid. Then we have the equation

$$\Phi_1 = MI_2 \ . \tag{26}$$

These methods are equivalent and give the same answer. We will illustrate both.

First let us imagine putting a current I_1 in the larger solenoid. Then there will be a constant magnetic field

$$B = n_1 \mu_0 I_1 \tag{27}$$

parallel to the direction of the solenoid and throughout the interior. This interior includes the smaller solenoid. If we restrict ourselves to a length ℓ of the small solenoid, which contains N_2 loops of wire, the magnetic flux is

$$\Phi_2 = \int \vec{B} \cdot d\vec{A} = N_2 B A_2 = N_2 n_1 \mu_0 I_1 \pi r_2^2 = \mu_0 n_1 n_2 \pi r_2^2 I_1 \ell \ . \tag{28}$$

Comparing to the defining relation for M, we find that the mutual inductance per length is

$$M/\ell = \mu_0 n_1 n_2 \pi r_2^2 \ . \tag{29}$$

Now we will compute the same thing using the other method. We let current I_2 run in the smaller loop. This generates a magnetic field

$$B = n_2 \mu_0 I_2 \tag{30}$$

parallel to the direction of the solenoid and throughout the interior (of the small solenoid). The magnetic field is zero outside of the small solenoid, so when we compute Φ_1 over a length ℓ of the big solenoid, the integral vanishes except over the cross-section of the small solenoid:

$$\Phi_1 = \int \vec{B} \cdot d\vec{A} = N_1 B A_2 = N_1 n_2 \mu_0 I_2 \pi r_2^2 = \mu_0 n_1 n_2 \pi r_2^2 I_2 \ell \ . \tag{31}$$

We find the same equation for M/ℓ .

Problem 4

a

The induced current will be proportional to the changing flux through the small ring, and will be in a direction such that the magnetic field it creates opposes the change in the external magnetic field. Since the current in the big loop is going clockwise and getting stronger, we know that the current in the small loop must be induced in a counterclockwise direction. Since the current in the big loop is changing linearly, its time derivative will be constant. Thus the current in the small loop should be constant over the range $0 < t < T$. So the graph should be a negative constant from $0 < t < T$.

b

Resistance is proportional to the length of the wire ($2\pi r$) and inversely proportional to the area:

$$R = \frac{2\pi r \rho_{\text{Cu}}}{a} \ . \tag{32}$$

c

The Biot-Savart rule for the magnetic field generated by a segment of wire $d\vec{\ell}$ with current I is

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{\ell} \times \hat{r}}{r^2} \ , \tag{33}$$

where $\vec{r}$ is a vector pointing from the line element to the place where the magnetic field is to be computed. In this case, the line elements are all tangent to the big circle and the vector $\vec{r}$ has length $10r$ and points to the center of the circle. The current, and hence the integral, goes clockwise around the circle. The right-hand-rule for the cross product tells us that the magnetic field points "into the page". The cross product does not contribute any $\sin\phi$ factors because the radial and tangent directions of the circle are perpendicular. To evaluate the integral we will write $d\ell = 10r d\theta$ and integrate the angle θ from 0 to 2π. For the magnitude of $\mathbf{B}$ we have

$$B = \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{I(10r d\theta)}{(10r)^2} = \frac{\mu_0 I}{20r} = \frac{3\mu_0 I_0}{40r} \ . \tag{34}$$

In the last equation we substituted $I = 1.5 I_0$.

d

Assuming the B field is uniform over the small circle, the magnetic flux is

$$\Phi(t) = \pi r^2 B(t) = \frac{\pi r \mu_0}{20} I(t). \tag{35}$$

The EMF is given by the derivative of the flux:

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\pi r \mu_0}{20} \frac{I_0}{T} \ . \tag{36}$$

The derivative of I is just I_0/T because I changes linearly by I_0 over time T. The result is independent of time, so it doesn't matter when we evaluate it. Ohm's Law gives us the current:

$$I_{\text{small ring}} = \mathcal{E}/R = -\frac{\pi r \mu_0}{20} \frac{I_0}{RT} \ . \tag{37}$$

The negative sign just means the current is running counterclockwise.

Problem 5

a

We begin by computing the electric field for $r > 4R$ by Gauss's Law. Our Gaussian surface will be a concentric cylinder of radius $r > 4R$ and length $\ell \ll L$. By symmetry the electric field can only point radially and the magnitude can only depend on the radius. So the "caps" of the Gaussian surface have zero electric flux, and the flux out of the side is easy to compute:

$$\oint \vec{E} \cdot d\vec{A} = E(r) \int_{\text{side}} dA = 2\pi r \ell E(r) \; . \tag{38}$$

The enclosed charge comes from both the inner cylinder and the outer cylinder. Our Gaussian surface encloses a fraction ℓ/L of the total charge of both surfaces, so $Q_{\text{enc}} = Q\ell/L$. Putting this together with Gauss's Law, we find

$$E(r) = \frac{Q_{\text{enc}}}{2\pi r \ell \epsilon_0} = \frac{Q}{2\pi r L \epsilon_0} \; . \tag{39}$$

Note that the net charge is positive, so the electric field points outward. That means the particle, which is negative, should be moving faster at the outer cylinder than where it starts. To compute the speed v_a, we simply add the work done by the electric field to the kinetic energy:

$$\frac{1}{2}Mv_a^2 = \frac{1}{2}Mv^2 + \int_{5R}^{4R} \vec{F} \cdot d\vec{r} = \frac{1}{2}Mv^2 - q\int_{5R}^{4R} \vec{E} \cdot d\vec{r} \tag{40}$$

$$= \frac{1}{2}Mv^2 - q\int_{5R}^{4R} \frac{Q}{2\pi r L \epsilon_0} dr = \frac{1}{2}Mv^2 + \frac{qQ}{2\pi L \epsilon_0} \ln\frac{5}{4} \; . \tag{41}$$

So the new speed is

$$v_a = \sqrt{v^2 + \frac{qQ}{M\pi L \epsilon_0} \ln\frac{5}{4}} \; . \tag{42}$$

b

We are going to repeat the method of part (a). We need to find the electric field for $R < r < 4R$ first. Again, we will use Gauss's Law for a concentric cylinder of radius r, this time with $R < r < 4R$. I will not repeat the analysis, but merely note that the charge enclosed is twice as large as it used to be because we are not enclosing the outer cylinder. So we get

$$E(r) = \frac{Q_{\text{enc}}}{2\pi r \ell \epsilon_0} = \frac{Q}{\pi r L \epsilon_0} \; . \tag{43}$$

Now we can compute the new speed v_b by finding the work done on the particle since it entered the outer cylinder:

$$\frac{1}{2}Mv_b^2 = \frac{1}{2}Mv_a^2 + \int_{4R}^{R} \vec{F} \cdot d\vec{r} = \frac{1}{2}Mv_a^2 - q\int_{4R}^{R} \vec{E} \cdot d\vec{r} \tag{44}$$

$$= \frac{1}{2}Mv_a^2 - q\int_{4R}^{R} \frac{Q}{\pi r L \epsilon_0} dr = \frac{1}{2}Mv_a^2 + \frac{qQ}{\pi L \epsilon_0} \ln 4 \; . \tag{45}$$

So we conclude

$$v_b = \sqrt{v_a^2 + \frac{2qQ}{M\pi L\epsilon_0}\ln 4}\ . \tag{46}$$

c

The speed v_a will be unchanged if a dielectric is added. There were only two things which determined the answer to part (a): symmetry and total charge. The dielectric will not change either of those if the entire dielectric is contained within our Gaussian surface (which it is).

d

v_b will be smaller in the presence of the dielectric, because the effective electric field in a dielectric is smaller than in vacuum, thus leading to a smaller amount of work done on the particle. Recall that in the microscopic picture of a dielectric in an electric field the surfaces of the dielectric become charged. This induced surface charge will partially cancel the charges on the cylinders. However, since the dielectric is electrically neutral overall, the total charge on both cylinders together does not change. That's why in part (c) we said that there was no change in v_a. But now we care about the effective charge on the inner cylinder by itself, which is lowered in the presence of the dielectric.