Fall 2009 - McKee - Midterm 2 solutions

Problem 1

The only meaningful way to compare the cost of the battery and the wall socket is cost per unit energy. In fact, cost per unit energy is the only information we are given about the expense of the wall socket. Cost per unit power doesn't make much sense, because power can be delivered over different lengths of time. The only way to compare cost per unit power is to specify that the power is delivered over a fixed length of time, but in that case, you are really comparing cost per unit energy anyway.

Many students wanted to assume that the wall socket was delivering the same current as the battery. There was no reason to make this assumption. Students who used this approach found that the wall socket was delivering 3 Watts, but then to complete the problem correctly they needed to relate this power to the cost per kilowatt hour, which would mean relating the 3 Watts to a quantity of energy.

Points were awarded according to the following rubric: At most 3 points were given for finding the power delivered by the battery.

$$P = IV = (25\text{ mA})(1.5\text{ V}) = 37.5\text{ mW} = 0.0375\text{ W}$$

At most 3 points were awarded for converting the power from the battery into the energy used over 820 hours:

$$\begin{aligned}\text{Energy} = \text{Power}\cdot\text{time} = (0.0375\text{ W})(820\text{ hours}) &= 30.75\text{ W h}\\ &= .03075\text{ kW h}\\ &(= 110700\text{ J})\\ &(= 110.7\text{ kJ})\end{aligned}$$

The conversion to Joules or kilo Joules was not necessary, but many students chose to make this conversion.

At most 2 points were awarded for finding the cost per unit energy for the battery, and then the final 2 points were awarded for finding the ratio of cost per unit energy. I did take off 2 points if students inverted the ratio they were asked to find (cost of battery to cost of wall socket).

$$\frac{\$1.70}{0.0375\text{ kW h}} = \frac{\$55.37}{\text{kW h}} \qquad \frac{\$1.70}{110700\text{ kJ}} = \frac{\$1.54\times10^{-5}}{\text{J}}$$

$$\frac{\frac{\$55.37}{\text{kW h}}}{\frac{\$0.10}{\text{kW h}}} = \frac{\frac{\$1.54\times10^{-5}}{\text{J}}}{\frac{\$2.78\times10^{-8}}{\text{J}}} = 550 \text{ times more expensive to use battery (2 sig figs)}$$

Problem 2

To simplify the problem, assume there is no charge on the wire. When the system reaches equilibrium, the potential on the two spheres will be equal. Assume charges Q_1 and Q_2 are on spheres with radii r_1 and r_2, respectively.
By charge conservation,

$$Q_1 + Q_2 = Q \tag{1}$$

Besides, the condition of equal potential gives

$$\frac{Q_1}{4\pi\epsilon_0 r_1} = \frac{Q_2}{4\pi\epsilon_0 r_2}$$

Or

$$Q_1 = \frac{r_1}{r_2}\, Q_2$$

Substitute the above equation into eq(1), we obtain

$$Q_2 = \frac{r_2}{r_1 + r_2} Q$$

Problem 3

[15] An electron is fired through a tiny hole into a capacitor at a velocity of 10^7 m s^{-1}. The capacitor is charged so that the field inside is $E = 2.6 \times 10^4$ V m^{-1}, pointing in the direction of the electron velocity. The capacitor plates are separated by a distance d such that the electron comes to rest just as it reaches the second plate. Find d.

- **Method 1 - Energy**. This method is easier and more straightforward, requiring no specialized kinematical formulas from Physics 7A.

  [1] The kinetic energy of the electron before it enters the capacitor is given by

  $$T = \frac{1}{2} m v^2.$$

  [3] The change in potential energy of a charged particle is related to the change in electric potential

  $$\Delta U = q \Delta V = -e \Delta V$$

  where $q = -e$ is the electron charge.

  [3] The electric potential of the electron once it has come to rest in the capacitor is given by the integral of the E-field over the path of the particle,

  $$\Delta V = -\int_0^d E\, dx = -Ed$$

  [2] Energy is conserved - all initial kinetic energy is converted to potential energy:

  $$T = U = \frac{1}{2} m v^2 = eEd$$

  [4] Correct algebra everywhere.

  $$d = \frac{m v^2}{2eE}$$

  [2] Correct numerical answer with correct units:

  $$d = 0.011 \text{ m} = 1.1 \text{ cm}$$

- **Method 2 - Kinematics**. This is the method most used to solve the problem, requiring recollection of a specialized kinematical formula from 7A. This method could not have been used to solve the problem if the E-field in the capacitor had been specified as non-uniform.

  [2] Recall Newton's second law

  $$F = ma.$$

[3] General relation for the force on a charged particle in an electric field

$$F = qE = -eE$$

where $q = -e$ is the electron charge.

[4] Kinematic relations from Physics 7A for constant acceleration:

$$v_f^2 = v_0^2 + 2a\Delta x$$

or

$$v_0 + a\Delta t = v_f \text{ and } \Delta x = \frac{1}{2}a\Delta t^2 + v_0\Delta t + x_0$$

[4] Correct algebra everywhere (lots more algebra for this method):

$$\begin{array}{c} v_f = 0 \quad v_0^2 = 2ad \\ a = eE/m \\ d = \frac{mv_0^2}{2eE} \end{array} \quad \text{or} \quad \begin{array}{c} v_f = 0 \quad v_0 = a\Delta t \\ a = eE/m \\ \Delta t = \frac{mv_0}{eE} \\ d = \frac{1}{2}a\Delta t^2 + v_0\Delta t = \frac{mv_0^2}{2eE} \end{array}$$

[2] Correct numerical answer with correct units:

$$d = 0.011 \text{ m} = 1.1 \text{ cm}$$

Problem 4 [15] - updated

There are few different ways to do this problem, the most straightforward being to use the formula for the E-field outside of a uniformly charged slab of thickness d that we computed in the homework:

Above the slab:

$$\oint \mathbf{E}\cdot\mathbf{dA} = \frac{Q}{\epsilon_0} \Rightarrow E(z)(2A) = \frac{\rho A d}{\epsilon_0} \Rightarrow \mathbf{E}(z) = \frac{\rho d}{2\epsilon_0}\hat{\mathbf{z}}$$

Inside the slab:

$$\oint \mathbf{E}\cdot\mathbf{dA} = \frac{Q}{\epsilon_0} \Rightarrow E(z)(2A) = \frac{\rho A(2z)}{\epsilon_0} \Rightarrow \mathbf{E}(z) = \frac{\rho z}{\epsilon_0}\hat{\mathbf{z}}$$

Below the slab:

$$\oint \mathbf{E}\cdot\mathbf{dA} = \frac{Q}{\epsilon_0} \Rightarrow E(z)(2A) = \frac{\rho A d}{\epsilon_0} \Rightarrow \mathbf{E}(z) = -\frac{\rho d}{2\epsilon_0}\hat{\mathbf{z}}$$

For the negative slab $\rho = -\rho$ in the above formula. Neither slab is centered around the y-z plane so we need to shift these equations: for the negative slab $z \to z + \frac{d}{2}$ and for the positive slab we take $z \to z - \frac{d}{2}$.

There are 4 regions of interest (in all cases $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$):

1. $z < -d$

$$\mathbf{E}(z) = -\frac{\rho d}{2\epsilon_0}\hat{\mathbf{z}} + \frac{\rho d}{2\epsilon_0}\hat{\mathbf{z}} = 0 \text{ [3pts]}$$

2. $z > d$

$$\mathbf{E}(z) = \frac{\rho d}{2\epsilon_0}\hat{\mathbf{z}} - \frac{\rho d}{2\epsilon_0}\hat{\mathbf{z}} \text{ [3pts]}$$

You CANNOT use Gauss' law to determine the E-field outside of the slabs by drawing a cylinder that passes through both slabs. Just because there is no charge inside does not mean that there is no E-field it simply means that the total flux sums to zero. I gave only 2/6 points if you did this; if you said the E-fields cancel but didn't ever say what they were I gave 4/6.

3. $-d < z < 0$

$$\mathbf{E}(z) = \frac{\rho}{\epsilon_0}\left[-\frac{d}{2} - \left(z + \frac{d}{2}\right)\right]\hat{\mathbf{z}} = -\frac{\rho(z+d)}{\epsilon_0}\hat{\mathbf{z}} \text{ [4.5pts]}$$

4. $0 < z < d$

$$\mathbf{E}(z) = \frac{\rho}{\epsilon_0}\left[\left(z - \frac{d}{2}\right) - \frac{d}{2}\right]\hat{\mathbf{z}} = \frac{\rho(z-d)}{\epsilon_0}\hat{\mathbf{z}} \text{ [4.5pts]}$$

I gave 3 points for the correct answer in each region (6/9) and 3 points for either computing the E-field inside a slab correctly, or for choosing a Gaussian surface that had one side outside the two slabs and one side inside the slabs, noting that the flux is EA not 2EA.

Problem 5

a

Let the energy stored in the first capacitor be U_1, and the energy stored in the second capacitor be U_2. The total energy stored in the equivalent capacitor we will call U_{eq}. In addition to the formula for energy stored in a capacitor, we need to know two facts about capacitors in series. First, by charge conservation, capacitors in series have the same charge Q, and this is also the charge of the equivalent capacitor.. Second, we need to know the formula for the equivalent capacitance of two capacitors in series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}\,. \tag{2}$$

Now it's just a matter of putting the formulas together:

$$U_{\text{eq}} = \frac{1}{2}\frac{Q^2}{C_{\text{eq}}} = \frac{1}{2}Q^2\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{1}{2}\frac{Q^2}{C_1} + \frac{1}{2}\frac{Q^2}{C_2} = U_1 + U_2\,. \tag{3}$$

The rubric for this part of the question is as follows (10 points total):

- 2 points for the energy formula in terms of Q
- 2 points for the equivalent capacitor formula for capacitors in series
- 4 points for realizing that all of the charges are equal in series
- 2 points for putting these things together into a logical argument

b

We have two scenarios, one where the capacitors are in series and another where they are in parallel. As discussed in part (a), for capacitors in series, the equivalent capacitance is

$$C_{\text{s}} = \frac{C_1 C_2}{C_1 + C_2}\,. \tag{4}$$

For capacitors in parallel, the equivalent capacitance is

$$C_{\mathrm{p}} = C_1 + C_2 \; . \tag{5}$$

By part (a), the total energy stored in each case is the same as the energy stored in the equivalent capacitor. Note that in both cases, the equivalent capacitor is connected with the same potential V_0. So we have two formulas relating U, C_1, and C_2:

$$U = \frac{1}{2} C_{\mathrm{s}} V_0^2 = \frac{1}{2} \frac{C_1 C_2}{C_1 + C_2} V_0^2 \; , \tag{6}$$

and

$$5U = \frac{1}{2} C_{\mathrm{p}} V_0^2 = \frac{1}{2} \left(C_1 + C_2\right) V_0^2 \; . \tag{7}$$

Dividing the second equation by the first gives

$$5 = \frac{\left(C_1 + C_2\right)^2}{C_1 C_2} = \frac{C_1}{C_2} \left(1 + \frac{C_2}{C_1}\right)^2 \; . \tag{8}$$

Let $x = C_2/C_1$. We can write the above equation as a quadratic equation for x:

$$x^2 - 3x + 1 = 0 \; , \tag{9}$$

which has the solution

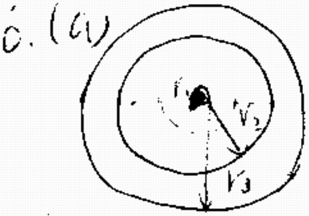
$$\boxed{x = \tfrac{3+\sqrt{5}}{2} \approx 2.62} \; . \tag{10}$$

Notice that we've taken the larger root because $C_2 > C_1$. The rubric for this part of the question is as follows (10 points total):

- 2 points for the equivalent capacitor formula for capacitors in parallel (the series formula was already in part (a))
- 2 points for the energy formula in terms of V_0
- 4 points for finding the equation for x
- 2 points for solving and selecting the correct root

Solution to Problem 6 Mckee

6. (a)



Gauss's law

2pt $$\vec{E}\cdot l\cdot 2\pi r = \frac{\lambda_1 \cdot l}{\varepsilon_0}$$ 1pt

$$\vec{E} = \vec{E}_\perp = \frac{\lambda_1}{2\pi\varepsilon_0 r}\hat{r}$$ 2pt

(b) take a point on the inner surface of the pipe as reference point.

2pt $$V(r) - V(r_2) = -\int_{r_2}^{r} \vec{E}\cdot d\vec{r} = -\int_{r_2}^{r} \frac{\lambda_1\, dr}{2\pi\varepsilon_0 r} = \underbrace{-\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r}{r_2}}_{1pt}$$

$$V(r) = V(r_2) + \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r_2}{r}$$ 2pt

(c)

$$q = E(r)\cdot 2\pi r\cdot l$$ 2pt

$$q = (\lambda_1 + \lambda_p)\cancel{2\pi r}\cdot l$$ 1pt

$$E\cdot 2\pi r\cdot l = \frac{(\lambda_1+\lambda_p)l}{\varepsilon_0}$$

$$\vec{E} = \frac{(\lambda_1+\lambda_p)}{2\pi\varepsilon_0}\frac{\hat{r}}{r}$$ 2pt

(d) conductor $\Rightarrow$ $V(r_2) = V(r_3)$ 2pt

1pt $$V(r) - V(r_3) = -\int_{r_3}^{r} \vec{E}\cdot d\vec{r} = \underbrace{-\int_{r_3}^{r} \frac{(\lambda_1+\lambda_p)}{2\pi\varepsilon_0 r}dr}_{1pt} = \underbrace{\frac{-(\lambda_1+\lambda_p)}{2\pi\varepsilon_0}\ln\frac{r}{r_3}}_{1pt}$$

$$V(r) = V(r_2) - \frac{(\lambda_1+\lambda_p)}{2\pi\varepsilon_0}\ln\frac{r}{r_3}$$ 1pt

(e) $\vec{E}(r_2 < r < r_3) = 0$ 1pt

$\lambda_1 + \lambda_2 = 0$ 2pt

$\lambda_3 = \lambda_p - \lambda_2 = \lambda_p + \lambda_1$ 2pt