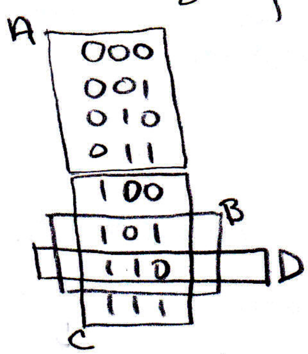
**Problem 1.1** *(48pts) True or False. Prove or show a counterexample:*

a. *12pts If $P(A) > P(B)$ and $P(C) > P(D)$ for events $A, B, C, D$, then $P(A \cap C) \geq P(B \cap D)$.*

FALSE

Consider the events as below, assuming all outcomes are equally probable:




$$P(A) = \frac{1}{2} > \frac{1}{4} = P(B)$$

$$P(C) = \frac{1}{2} > \frac{1}{8} = P(D)$$

$$P(A \cap C) = 0 \not\geq \frac{1}{8} = P(B \cap D)$$

b. *12pts. If $X, Y$ are both Bernoulli random variables (i.e. they can only take on the values 0 and 1) and $E[XY] = E[X]E[Y]$, then they are independent of each other.*

TRUE

Let $P(X=1) = p = P(X=1, Y=0) + P(X=1, Y=1)$

$P(Y=1) = q = P(X=0, Y=1) + P(X=1, Y=1)$

with joint density $P_{X,Y}(x,y)$

Then $\mathbb{E}[XY] = \sum_x \sum_y xy\, P_{xy}(x,y) = P(X=1, Y=1)$

$\mathbb{E}[X]\mathbb{E}[Y] = pq$

If $pq = P(X=1, Y=1)$, Then we need to prove in general $P_{XY}(x,y) = P_X(x) P_Y(y)$

Notice That with normalization $\left(\sum_x \sum_y P_{XY}(x,y) = 1\right)$, we have 4 eqns and 4 unknowns defined by The system

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} P(X=0, Y=0) \\ P(X=0, Y=1) \\ P(X=1, Y=0) \\ P(X=1, Y=1) \end{bmatrix} = \begin{bmatrix} p \\ q \\ pq \\ 1 \end{bmatrix}$$

Since The matrix is invertible, There is a unique solution.

$\Rightarrow P(X=0, Y=0) = -p - q + pq + 1 = (1-p)(1-q) = P(X=0)P(Y=0)$

$P(X=0, Y=1) = (1-p)\, q = P(X=0)P(Y=1)$

$P(X=1, Y=0) = p(1-q) = P(X=1)P(Y=0)$

$P(X=1, Y=1) = pq = P(X=1)P(Y=1)$

$\therefore X, Y$ are independent If $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$

c. 12pts *Let $X$ be a continuous non-negative (i.e. $P(X < 0) = 0$) random variable with density $f_X(t)$ that satisfies $0 \le f_X(t) \le 1$ for every $t$. Then a median (value $m$ at which $P(X \le m) = P(X \ge m)$) of $X$ must be larger than $\frac{1}{3}$.*

TRUE

If $0 \le f_X(t) \le 1$,

then $\max_{0 \le f_X(t) \le 1} P(X \le \frac{1}{3}) = \frac{1}{3}(1)$ since $P(X<0)=0$

$= \frac{1}{3}$

$\therefore P(X \le \frac{1}{3}) < \frac{1}{2}$ always

and so the median MUST be greater than $\frac{1}{3}$.

d. 12pts *The number of distinct ways that $k$ identical balls can be arranged into $n$ distinct bins is $\frac{n!}{(n-k)!k!}$.*

FALSE

Take the example 2 balls in 3 bins:

The balls can be arranged in the following ways

| Bin 1 | Bin 2 | Bin 3 |
|---|---|---|
| X X | | |
| | X X | |
| | | X X |
| X | X | |
| | X | X |
| X | | X |

So there are 6 possibilities

but $\frac{3!}{1!\,2!} = 3$.

**Problem 1.2** *(70pts) (The parts of this problem are largely independent of each other.)*

*Mary Jo Lisa is looking for someone to marry. This is the way that matches are made in this completely make-believe society (for the first few parts): she is paired up with a random person and can observe them. At the end of that observation, both her and the other person vote "Yes" or "No." If both vote "Yes," they get married and the process stops. If either votes "No," she gets paired up with another random person and the process repeats.*

a. *10 pts Suppose that this society is governed entirely by Horoscopes. People are born randomly and uniformly into the 12 signs of the Zodiac: arranged in their traditional circular pattern. People will say "Yes" to anyone who either has the same sign or either of the two immediately adjacent signs.* **What is the expected number of people that Mary will meet in this match-making process?**

P( any random person will be a match for Mary ) $= \frac{3}{12} = \frac{1}{4}$

P( Mary meets match on person k )

$= P($ people $1 \ldots k-1$ not matches, person k is a match $)$

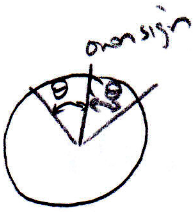
$= \left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{4}\right)$

This is a GEOMETRIC R.V. with success probability $\frac{1}{4}$

So

$$\boxed{\mathbb{E}[\text{number of people Mary must meet}] = \frac{1}{\frac{1}{4}} = 4}$$

b. 10 pts *Suppose that instead of having 12 discrete signs of the Zodiac, the people believe in continuous signs according to the exact angle from 0 to $2\pi$ that the Sun marks on the Zodiac circle at their time of birth. (Assume this is uniform and independent across people) They are willing to marry anyone within $\pm\theta$ radians of their own sign.* **As a function of $\theta$, what is the probability that Mary will end up meeting more than 3 people in the course of this process?**




$$P(\text{random person is a match}) = \frac{2\theta}{2\pi} = \frac{\theta}{\pi}$$

$$P(\text{Mary meets match on person } k) = \left(1-\frac{\theta}{\pi}\right)^{k-1}\left(\frac{\theta}{\pi}\right)$$

This again is GEOMETRIC R.V.

So

$P(\text{meet more than 3 people})$

$$= \sum_{k=4}^{\infty}\left(1-\frac{\theta}{\pi}\right)^{k-1}\left(\frac{\theta}{\pi}\right)$$

$$= 1 - \sum_{k=1}^{3}\left(1-\frac{\theta}{\pi}\right)^{k-1}\left(\frac{\theta}{\pi}\right)$$

$$= \boxed{1 - \left[\frac{\theta}{\pi} + \left(1-\frac{\theta}{\pi}\right)\frac{\theta}{\pi} + \left(1-\frac{\theta}{\pi}\right)^2\left(\frac{\theta}{\pi}\right)\right]}$$

Alternatively:

$P(\text{meet} > 3 \text{ people})$

$$= \sum_{k=4}^{\infty}\left(1-\frac{\theta}{\pi}\right)^{k-1}\left(\frac{\theta}{\pi}\right)$$

$$= \frac{\theta}{\pi}\sum_{k=4}^{\infty}\left(1-\frac{\theta}{\pi}\right)^{k-1}$$

$$= \frac{\theta}{\pi}\left(1-\frac{\theta}{\pi}\right)^3\sum_{k=0}^{\infty}\left(1-\frac{\theta}{\pi}\right)^k$$

$$= \frac{\theta}{\pi}\left(1-\frac{\theta}{\pi}\right)^3\left(\frac{1}{1-\left(1-\frac{\theta}{\pi}\right)}\right)$$

$$= \frac{\theta}{\pi}\left(1-\frac{\theta}{\pi}\right)^3\left(\frac{1}{\frac{\theta}{\pi}}\right)$$

$$= \left(1-\frac{\theta}{\pi}\right)^3$$



Note that this is true only for $\theta < \pi$. If $\theta \geq \pi$, then ANY other person would be a match for Mary.

So if $\theta \geq \pi$, $P(\text{Mary meets match on person } 1) = 1$

$P(\text{Mary meets} > 3 \text{ people}) = 0.$

c. 15 pts *Suspose now that instead there are two kinds of potential partners out there: $\frac{1}{3}$ H-type ("Hot") and $\frac{2}{3}$ N-type ("Not"). Mary is H-type. For H-types, the probability that a random H-type is a good match to them is $\frac{3}{4}$. The probability that an N-type is a match is a mere $\frac{3}{40}$. Given that her candidates will be drawn randomly from the population,* **what is the expected number of people that Mary will meet in this match-making process? And what is the probability that she will end up married to a fellow H-type?**

$$P(H) = \frac{1}{3} \qquad P(\text{match} \mid H) = \frac{3}{4}$$

$$P(N) = \frac{2}{3} \qquad P(\text{match} \mid N) = \frac{3}{40}$$

$$\begin{aligned} P(\text{random person is a match}) &= P(\text{random person match} \mid H)P(H) \\ &\quad + P(\text{random person match} \mid N)P(N) \\ &= \frac{3}{4}\left(\frac{1}{3}\right) + \left(\frac{3}{40}\right)\left(\frac{2}{3}\right) \\ &= \frac{1}{4} + \frac{1}{20} \\ &= \frac{6}{20} = \frac{3}{10} \end{aligned}$$

So, again, This is a geometric with success probability $\frac{3}{10}$.

$$\boxed{E[\text{number of people Mary meets}] = \frac{1}{\frac{3}{10}} = \frac{10}{3}}$$

$$\begin{aligned} P(\text{Mary gets married to an 'H'}) &= P(H \mid \text{match}) \\ &= \frac{P(\text{match} \mid H)P(H)}{P(\text{match} \mid H)P(H) + P(\text{match} \mid N)P(N)} \end{aligned}$$



$$= \frac{\frac{3}{4}\left(\frac{1}{3}\right)}{\left(\frac{3}{4}\right)\left(\frac{1}{3}\right) + \left(\frac{3}{40}\right)\left(\frac{2}{3}\right)}$$

$$\boxed{P(\text{married to an 'H'}) = \frac{\frac{1}{4}}{\frac{3}{10}} = \frac{5}{6}}$$

d. 20 pts *(builds on the assumptions in part c) Suppose now that the observations are imperfect. Mary knows that she is an H-type, but she cannot see the underlying compatibility. She only knows the background proportions of the two types. She cannot even perfectly see the type of the person she meets. Instead, she gets a noisy observation that is symmetrically wrong with probability $p < \frac{1}{2}$ (i.e. The probability that she thinks the person is an H given that they are actually an N is p and similarly for N's seen as H's.)* and so does the other person *who also secretly knows their true type.*

*The other person will say "Yes" if their noisy observation matches their own true type. Similarly, Mary will say "Yes" if her noisy observation says the person is an H. If both say "Yes," they'll get married.*

*As a function of p,* **what is the expected number of people that Mary will meet in this match-making process? And what is the probability that she will end up unhappily married? Given that she is unhappy, what is the probability that she is married to a fellow H?**

*(You do not have to simplify these expressions too much)*

From (c), we have: $P(H) = \frac{1}{3}$, $P(\text{match} \mid H) = \frac{3}{4}$

$P(N) = \frac{2}{3}$, $P(\text{match} \mid N) = \frac{3}{40}$

New for (d): $P(\text{sees } H \mid \text{actually } H) = 1-p$ $\quad P(\text{sees } H \mid \text{actually } N) = p$

$P(\text{sees } N \mid \text{actually } N) = 1-p$ $\quad P(\text{sees } N \mid \text{actually } H) = p$

Now we need the probability of a marriage:

$P(\text{random person marriage}) = P(\text{mary sees own type AND other person sees own type})$

$= P(\text{Mary sees } H, \text{ O.P sees } H \text{ and is } H) + P(\text{Mary sees } H, \text{ O.P sees } N \text{ and is } N)$

$= P(\text{Mary sees } H \mid \text{O.P. actually } H) P(\text{O.P. is } H) P(\text{O.P. sees } H \mid \text{Mary is } H)$
$+ P(\text{Mary sees } H \mid \text{O.P. actually } N) P(\text{O.P is } N) P(\text{O.P. sees } N \mid \text{Mary is } H)$



$= (1-p)(1-p)\frac{1}{3} + p \cdot p \left(\frac{2}{3}\right)$

$= \frac{1}{3}(1-p)^2 + \frac{2}{3}p^2$

$= \frac{1}{3}(1-2p+p^2) + \frac{2}{3}p^2$

$= \frac{1}{3} - \frac{2}{3}p + p^2$

*Scratch page*

So,

$$P(\text{Mary gets married to person } k) = \left(1-\tfrac{1}{3}+\tfrac{2}{3}p-p^2\right)^{k-1}\left(\tfrac{1}{3}-\tfrac{2}{3}p+p^2\right)$$

This is again a GEOMETRIC R.V.

So

$$\boxed{E[\text{number of people Mary will meet}] = \frac{1}{\frac{1}{3}-\frac{2}{3}p+p^2}}$$

Now, we want

$$P(\text{Mary will be unhappily married}) = P(\text{bad match} \mid \text{Mary married on } H)P(\text{married on } H) + P(\text{bad match} \mid \text{married on } N)P(\text{married } N)$$

So we need

$$P(\text{married on } H) = P(\text{'H'} \mid \text{married})$$

$$= \frac{P(\text{marriage} \mid H)P(H)}{P(\text{marriage} \mid H)P(H) + P(\text{marriage} \mid N)P(N)}$$

$$= \frac{P(\text{Mary sees } H \mid \text{O.P is } H)P(\text{O.P. sees } H \mid \text{Mary } H)P(\text{O.P. } H)}{P(\text{random person married (from previous page)})}$$

$$= \frac{(1-p)^2\frac{1}{3}}{\frac{1}{3}-\frac{2}{3}p+p^2}$$

$$P(\text{married on } N) = \frac{P(\text{marriage} \mid N)P(N)}{P(\text{random person marriage})}$$

$$= \frac{p^2\frac{2}{3}}{\frac{1}{3}-\frac{2}{3}p+p^2}$$

So:

$$\boxed{P(\text{unhappily married}) = \frac{1}{4}\left(\frac{(1-p)^2\frac{1}{3}}{\frac{1}{3}-\frac{2}{3}p+p^2}\right) + \frac{37}{40}\left(\frac{\frac{2}{3}p^2}{\frac{1}{3}-\frac{2}{3}p+p^2}\right)}$$

(2.d cont)

Given she is unhappily married, what is the probability of being married to a fellow 'H'?

$$P(\text{marries 'H'} \mid \text{unhappy}) = \frac{P(\text{bad match} \mid \text{marries an 'H'})\,P(\text{marrying an H})}{P(\text{bad match} \mid \text{marries H})\,P(\text{marries H}) + P(\text{bad match} \mid \text{marries N})\,P(\text{marries N})}$$

$$P(\text{marries H} \mid \text{unhappy}) = \frac{\frac{1}{4}\left(\frac{\frac{1}{3}(1-p)^2}{\frac{1}{3}-\frac{2}{3}p+p^2}\right)}{\frac{1}{4}\left(\frac{\frac{1}{3}(1-p)^2}{\frac{1}{3}-\frac{2}{3}p+p^2}\right) + \frac{37}{40}\left(\frac{\frac{2}{3}p^2}{\frac{1}{3}-\frac{2}{3}p+p^2}\right)}$$

where all expressions have been calculated on the last page.

$$\boxed{P(\text{marries H} \mid \text{unhappy}) = \frac{\frac{1}{12}(1-p)^2}{\frac{1}{12}(1-p)^2 + \frac{37}{40}\left(\frac{2}{3}p^2\right)}}$$

e. 20 pts *In another society, people are matched up in a completely different way. Single men are ordered alphabetically by their names and marked with their number in the list (So 1 is the first person in alphabetical order and so on). Single women are also ordered alphabetically by their names and assigned their number in the list. (Assume that this society has a lot of distinct names and so there are no two people with the same name)*

*Then the two groups are randomly permuted and paired off until there are no more left in at least one group. If a man's number is not smaller than the woman's number, then they get married. Given that we start with 2n men and n women,* **what is the expected number of marriages that will occur?**

It is easiest to do this problem from the female perspective, in which case this is the same as leaving the females ordered and just permuting the men.

Let $X_i = \begin{cases} 1 & \text{if female } i \text{ gets married} \\ 0 & \text{ow.} \end{cases}$

If we randomly permute the men, then for each female $i$, she is grouped with a man whose number is uniformly distributed over the set $\{1, \ldots, 2n\}$

Let $Y \sim \text{unif}\{1 \ldots 2n\}$ be the number of the man female $i$ is paired with.

So for female $i$:

$$
\begin{aligned}
P(X_i = 1) &= P(Y \geq i) \\
&= 1 - P(Y < i) \\
&= 1 - \frac{i-1}{2n} \\
&= \frac{2n - i + 1}{2n}
\end{aligned}
$$

Then

$$
\mathbb{E}[\text{number of marriages}] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} P(X_i = 1)
$$

$$
= \sum_{i=1}^{n} \frac{2n+1}{2n} - \frac{i}{2n} = \left(\frac{2n+1}{2n}\right)n - \frac{(n+1)n}{2}\left(\frac{1}{2n}\right)
$$

$$
= \frac{4n^2 + 2n - n^2 - n}{4n} = \boxed{\frac{3n+1}{4}}
$$