SOLUTIONS

Fall 2009: EECS126 Midterm 2

No Collaboration Permitted. One sheet of notes is permitted. Turn in with your exam.

Be clear and precise in your answers

Write your name and student ID number on every sheet.

Come to the front if you have a question.

80-90 points is a very good score. There are more points than that on the exam. So look over the entire exam. Some points at the end may be easier to get than others earlier in the exam.

EE 126 Midterm 2 Soln

Problem 1.1 *(36pts) True or False. Prove or show a counterexample:*

a. *12pts Let X be a continuous non-negative (i.e. $P(X<0)=0$) random variable with density $f_X(t)$ that satisfies $0 \le f_X(t) \le 1$ for every t. Then a median (value m at which $P(X \le m) = P(X \ge m)$) of X must be larger than $\frac{1}{3}$.*

$\boxed{\text{TRUE}}$

If $f_X(t) = 0$ for $t < 0$

and $0 \le f_X(t) \le 1$ for $t > 0$,

Then the maximum density below $\frac{1}{3}$ is $\int_0^{\frac{1}{3}} f_X(t)dt = \int_0^{\frac{1}{3}} 1\, dt = \frac{1}{3}$.

$\therefore$ The point m at which $\int_0^m f_X(t)dt = \frac{1}{2}$

MUST be greater than $\frac{1}{3}$.

b. *12pts. If X has marginal PDF $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, and Y has the same marginal PDF, then $E[XY] = 0$ implies that X and Y are independent.*

$\boxed{\text{FALSE}}$

$X \sim N(0,1)$ which has marginal PDF $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Let $Z = \begin{cases} 1 & \text{wp } \frac{1}{2} \\ -1 & \text{wp } \frac{1}{2} \end{cases}$

and let $Y = XZ$.

Then $f_Y(y) = f_{Y|Z=-1}(y|Z=-1)P(Z=-1) + f_{Y|Z=1}(y|Z=1)P(Z=1)$

$= f_{-X}(y)\frac{1}{2} + f_X(y)\frac{1}{2}$ (Total Law of Prob.)

$= f_X(y)\frac{1}{2} + f_X(y)\frac{1}{2}$ (Symmetry of Gaussian)

$= f_X(y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

And $\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY|Z]] = \mathbb{E}[XY|Z=-1]\frac{1}{2} + \mathbb{E}[XY|Z=1]\frac{1}{2}$

$= \mathbb{E}[-X^2]\frac{1}{2} + \mathbb{E}[X^2]\frac{1}{2} = \frac{1}{2}\left[\mathbb{E}[X^2] - \mathbb{E}[X^2]\right] = 0$

But, X and Y are NOT independent since if $X=a$, Y can take only values a or $-a$.

c. *12pts If X_1, X_2, X_3 are identically distributed, have finite means $E[X]$ and variances $Var[X]$, but are not independent, then there must exist some constant $a > 0$ for which $P(|\frac{X_1+X_2+X_3}{3} - E[X]| \geq a) > \frac{Var[X]}{3a^2}$.*

$\boxed{\text{FALSE}}$ We'll show this by finding an expression that determines the properties on X_1 X_2 and X_3 for which $P(|\frac{X_1+X_2+X_3}{3} - \mathbb{E}[X]| \geq a) \leq \frac{var(X)}{3a^2}$ holds for any a.
and then build X_1 X_2 X_3 with all the necessary properties

Chebyshev's inequality tells us

$$P(|Z - \mathbb{E}[Z]| \geq a) \leq \frac{var(Z)}{a^2}$$

Let $Z = (X_1 + X_2 + X_3)\frac{1}{3}$ for X_1 X_2 X_3 identically distributed but not independent.

Then $\mathbb{E}[Z] = \frac{1}{3}\mathbb{E}[X_1+X_2+X_3] = \frac{1}{3}(\mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3]) = \mathbb{E}[X]$

$$\begin{aligned} var(Z) &= \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 \\ &= \mathbb{E}\left[\left(\frac{X_1+X_2+X_3}{3}\right)^2\right] - \mathbb{E}[X]^2 \\ &= \frac{1}{9}\left(\mathbb{E}[X_1^2] + \mathbb{E}[X_2^2] + \mathbb{E}[X_3^2] + 2\mathbb{E}[X_1X_2] + 2\mathbb{E}[X_1X_3] + 2\mathbb{E}[X_2X_3]\right) - \mathbb{E}[X]^2 \\ &= \frac{1}{3}\mathbb{E}[X^2] - \frac{1}{3}(\mathbb{E}[X])^2 + \frac{2}{9}(\mathbb{E}[X_1X_2] + \mathbb{E}[X_1X_3] + \mathbb{E}[X_2X_3]) - \frac{2}{3}\mathbb{E}[X]^2 \\ &= \frac{var(X)}{3} + \frac{2}{9}(\mathbb{E}[X_1X_2] + \mathbb{E}[X_1X_3] + \mathbb{E}[X_2X_3]) - \frac{2}{3}\mathbb{E}[X]^2 \end{aligned}$$

So, $P(|\frac{X_1+X_2+X_3}{3} - \mathbb{E}[X]| \geq a) \leq \frac{var(X)}{3a^2}$ for any a as long as

$$\frac{2}{9}(\mathbb{E}[X_1X_2] + \mathbb{E}[X_1X_3] + \mathbb{E}[X_2X_3]) - \frac{2}{3}\mathbb{E}[X]^2 = 0$$

This happens if X_1 X_2 and X_3 are pairwise independent, or pairwise uncorrelated,
which can happen even if X_1 X_2 and X_3 are <u>NOT</u> independent.

There is an example on the next page.

(1.1.c cont)

Example:

Let $X_1, X_2, X_3 \sim \text{Bern}(\frac{1}{2})$ marginal distribution

$X_1 = X_2 \oplus X_3$, X_2 indep of X_3

Obviously these are not independent -- given any 2 of X_1, X_2 and X_3, the third is deterministic.

But, they are pairwise independent

X_2 and X_3 independent by definition

$P(X_1 = 1 | X_2 = 1) = P(X_3 = 0 | X_2 = 1) = \frac{1}{2} = P(X_1 = 1 | X_2 = 0)$

So X_1 and X_2 are independent.

$P(X_1 = 1 | X_3 = 1) = P(X_2 = 0 | X_3 = 1) = \frac{1}{2} = P(X_1 = 1 | X_3 = 0$

So X_1 and X_3 are independent.

Therefore, plugging into our condition from the last page:

$$\frac{2}{9}(\mathbb{E}[X_1X_2] + \mathbb{E}[X_2X_3] + \mathbb{E}[X_1X_3]) - \frac{2}{3}\mathbb{E}[X]^2$$

$$= \frac{2}{9}(\mathbb{E}[X_1]\mathbb{E}[X_2] + \mathbb{E}[X_2]\mathbb{E}[X_3] + \mathbb{E}[X_1]\mathbb{E}[X_3]) - \frac{2}{3}\mathbb{E}[X]^2$$

$$= \frac{2}{9}[3\,\mathbb{E}[X]^2] - \frac{2}{3}\mathbb{E}[X]^2$$

$$= 0$$

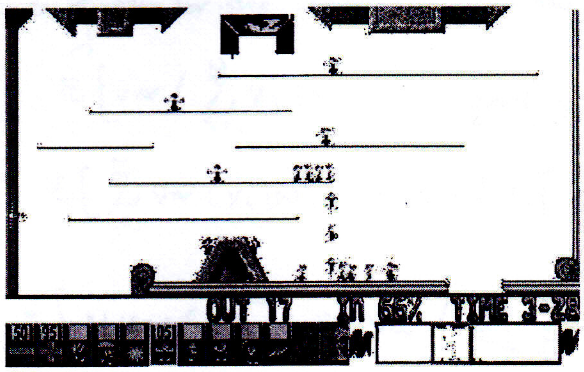
$\therefore \text{var}(Z) = \frac{\text{var}(X)}{3}$ for $Z = \frac{X_1 + X_2 + X_3}{3}$

and

$$P\left(\left|\frac{X_1 + X_2 + X_3}{3} - \mathbb{E}[X]\right| \geq a\right) \leq \frac{\text{var}(X)}{3a^2} \text{ for any } a$$

and therefore the statement given is FALSE.

Problem 1.2 *(80 pts) Video-game-world lemmings are very loyal creatures, but unfortunately are very* **stupid**, *and often forget what they are doing. You have a few tasks to complete in a new video-game and only an army of these lemmings to help you... (Each problem part stands apart from the others.)*



a. *15 pts You need to use your lemmings to help build a bridge.*

i. *To get to the bridge, the lemmings have to walk through a door that will slam shut (with i.i.d. probability p) after any lemming passes through. What is the distribution of the number of lemmings, N, that make it through the door?*

$\mathbb{E}[X_i] = \frac{1}{\lambda}$

$var(X_i) = \frac{1}{\lambda^2}$

ii. *Each of the N lemmings from part (i) will independently contribute a length $X_i \sim Exp(\lambda)$ to the bridge before forgetting what it is doing and wandering off. What is the expected length of the bridge built before all your lemmings wander away?*

iii. *What is the variance of the bridge length?*

(i) Each lemming has a probability p of being the last lemming through the door. So, this is a geometric r.v.:

$N \sim Geom(p)$ $\quad f_N(n) = (1-p)^{n-1} p$

$\mathbb{E}[N] = \frac{1}{p} \quad var(N) = \frac{1-p}{p^2}$.

(ii) the expected length of the bridge is the expected value of a random number of random variables, so we use the law of iterated expectation:

$$\mathbb{E}[\text{length of bridge}] = \mathbb{E}\left[\sum_{i=1}^{N} X_i\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N} X_i \mid N\right]\right] \quad \text{(iterated expectation)}$$

$$= \mathbb{E}\left[\sum_{i=1}^{N} \mathbb{E}[X_i \mid N]\right] \quad \text{(linearity)}$$

$$= \mathbb{E}\left[\sum_{i=1}^{N} \mathbb{E}[X_i]\right] \quad \text{(iid } X_i\text{'s, indep of } N)$$

$$= \mathbb{E}[N\,\mathbb{E}[X_1]]$$

$$= \mathbb{E}[N]\,\mathbb{E}[X_1] \quad \mathbb{E}[X_1] \text{ is just a scalar.}$$

$$= \frac{1}{\lambda p}$$

Extra page

(Problem 1.2 a cont)

(iii) For the variance of the bridge length, we need the law of total variance:

$$
\begin{aligned}
\mathrm{var}\left(\sum_{i=1}^{N} X_i\right) &= \mathbb{E}\left[\mathrm{var}\left(\sum_{i=1}^{N} X_i \mid N\right)\right] + \mathrm{var}\left(\mathbb{E}\left[\sum_{i=1}^{N} X_i \mid N\right]\right) && \text{Law of total variance} \\
&= \mathbb{E}\left[\sum_{i=1}^{N} \mathrm{var}(X_i \mid N)\right] + \mathrm{var}\left(\mathbb{E}\left[\sum_{i=1}^{N} X_i \mid N\right]\right) && \mathrm{var}\left(\textstyle\sum Y_i\right) = \textstyle\sum \mathrm{var}(Y_i) \text{ for } Y_i \text{ iid.} \\
&= \mathbb{E}\left[N\,\mathrm{var}(X_1)\right] + \mathrm{var}\left(\mathbb{E}\left[\sum_{i=1}^{N} X_i \mid N\right]\right) && X_i \text{ iid.} \\
&= \mathbb{E}[N]\,\mathrm{var}(X_1) + \mathrm{var}\left(\mathbb{E}\left[\sum_{i=1}^{N} X_i \mid N\right]\right) && \mathrm{var}(X_1) \text{ is a scalar.} \\
&= \mathbb{E}[N]\,\mathrm{var}(X_1) + \mathrm{var}\left(\sum_{i=1}^{N} \mathbb{E}[X_i \mid N]\right) && \text{Linearity of } \mathbb{E} \\
&= \mathbb{E}[N]\,\mathrm{var}(X_1) + \mathrm{var}\left(N\,\mathbb{E}[X_1]\right) && X_i \text{ iid} \\
&= \mathbb{E}[N]\,\mathrm{var}(X_1) + (\mathbb{E}[X_1])^2\,\mathrm{var}(N) && \mathbb{E}[X_1] \text{ is just a scalar} \\
&= \frac{1}{p}\left(\frac{1}{\lambda^2}\right) + \left(\frac{1}{\lambda}\right)^2\left(\frac{1-p}{p^2}\right) \\
&= \frac{p+1-p}{p^2\lambda^2} \\
&= \frac{1}{p^2\lambda^2}
\end{aligned}
$$

b. 15 pts *Each lemming can jump either a long distance (with probability p) or a short distance (with probability $1-p$). The long-jumping lemmings will jump a distance $X_l \sim Exp(\lambda)$ and the short-jumping lemmings will jump a distance $X_s \sim Exp(2\lambda)$.*

You have two randomly drawn independent lemmings and want them to cross a chasm x wide. What is the probability that both will jump across the chasm safely? (Leave this as an expression involving p and λ, but don't leave integrals in your final answer)

Let the distances jumped by your 2 drawn lemmings be Y_1 and Y_2.

Then

$$P(\text{both jump across safely}) = P(Y_1 > x, Y_2 > x)$$

$$= P(Y_1 > x) P(Y_2 > x) \quad \text{by independence.}$$

$$= [P(Y_1 > x)]^2 \quad \text{since they are identically distributed.}$$

$$= [P(Y_1 > x \mid Y_1 \text{ is long jumping}) P(Y_1 \text{ is long jumping}) + P(Y_1 > x \mid Y_1 \text{ is short jumping}) P(Y_1 \text{ is short jumping})]^2$$

by law of total probability

$$= [\, p \cdot P(Exp(\lambda) > x) + (1-p) P(Exp(2\lambda) > x)]^2$$

note: $P(Exp(\lambda) < x) = 1 - e^{-\lambda x}$ (CDF)

so $P(Exp(\lambda) > x) = 1 - P(Exp(\lambda) < x)$
$= e^{-\lambda x}$

$$= (pe^{-\lambda x} + (1-p)e^{-2\lambda x})^2$$

c. 10 pts *A fixed number n of your lemmings are walking toward a cliff. Each lemming will randomly trip (independently of the others) with probability $1-p$. If it trips, a lemming will turn around and walk away from the cliff and be safe. Otherwise, it will fall off the cliff.*

Using a Chernoff bound, bound the probability that you will lose more than $(p+\epsilon)n$ lemmings over the side of the cliff.

Number the lemmings 1 to n.

Let $X_i = \begin{cases} 1 & \text{if lemming } i \text{ falls of the cliff} \\ 0 & \text{ow.} \end{cases}$

So $X_i \sim \text{Bern}(p)$, independent of the other lemmings.

Then, you want

$$P\left(\sum_{i=1}^{n} X_i > (p+\varepsilon)n\right) = P\left(\frac{1}{n}\sum_{i=1}^{n} X_i > p+\varepsilon\right)$$

$$\leq e^{-nD(p+\varepsilon \| p)}$$

by chernoff bound

where $D(p+\varepsilon\|p) = (p+\varepsilon)\ln\left(\frac{p+\varepsilon}{p}\right) + (1-p-\varepsilon)\ln\left(\frac{1-p-\varepsilon}{1-p}\right)$

<u>NOTE</u>: This form of the Chernoff bound with

$\Phi_{X_i}(a) = D(a \| E[X_i])$

is true only if $X_i \sim \text{Bernoulli}(p)$.

Also, it is not necessary to do the Taylor approx for this quantity. We did that in class to get some intuition that something like the CLT must exist.

Once you put in an approx like the Taylor approx, best you can claim is an <u>approximate</u> bound. It is no longer a strict bound!

d. 25 pts You are looking for new lemmings. Lemmings all come from the same hometown which is at an unknown location $X \sim Unif[0, 100]$ from where you are currently standing (you are living in a 1-D world). You tend to find a lemming once it has had a chance to wander $W \sim Unif[0, 20]$ from where it started. Each lemming wanders independently of other lemmings and independently of where it started.

You find a lemming at a point $Y = y$.

i. What are the MMSE and LLSE estimates of the location of the lemmings' hometown?

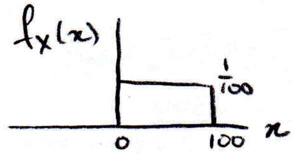
ii. Answer part (i) again except assuming $X \sim N(0, 100)$ and $W \sim N(0, 20)$?

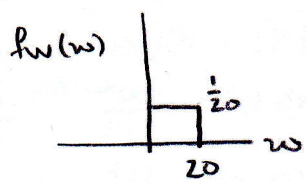
iii. What is the mean squared error in each of the estimates of parts (i) and (ii)?

iv. For the Gaussian case of (ii), you find a second lemming at a point y_2. What is your new estimate of the location of the lemmings' hometown?

(i) MMSE:

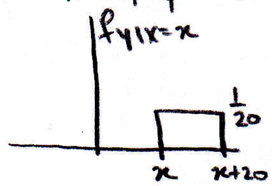
We have



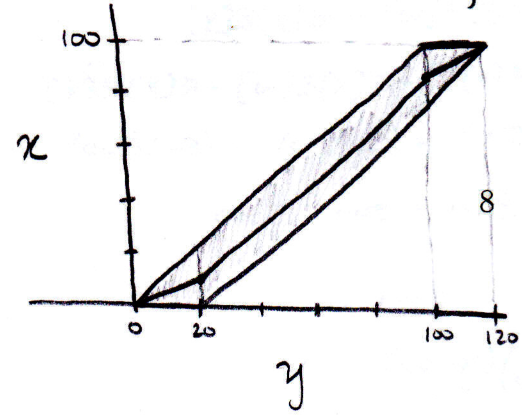


and $Y = X + W$

So for a given $X = x$, Y has the distribution



And the joint distribution $f_{X,Y}(x,y)$ lies on this support set:



and since given $X = x$, Y is uniform and given $Y = y$, X is uniform, $f_{X,Y}(x,y)$ is uniform over this support set.

So, $E[X|Y=y]$ = midpoint of the range of X when $Y = y$. This is drawn to the left and the expression is given on the next page.

(1.2.d.i cont)

MMSE expression:

$$\mathbb{E}[X|Y=y] = \begin{cases} \frac{1}{2}y, & 0 \le y \le 20 \\ y-10, & 20 \le y \le 100 \\ \frac{1}{2}y+40, & 100 \le y \le 120 \end{cases}$$

LLSE (uniform case)

For the LLSE, we have a formula to use:

$$LLSE[X|Y=y] = \mathbb{E}[X] + \frac{\mathrm{Cov}(X,Y)}{\mathrm{var}(Y)}(y - \mathbb{E}[Y])$$

so we need to calculate pieces

$$\mathbb{E}[X] = \mathbb{E}[\mathrm{unif}(0,100)] = 50$$

$$\mathbb{E}[W] = \mathbb{E}[\mathrm{unif}(0,20)] = 10$$

$$\mathbb{E}[Y] = \mathbb{E}[X+W] = \mathbb{E}[X] + \mathbb{E}[W] = 60$$

$$\mathrm{var}(X) = \frac{100^2}{12} = \frac{10000}{12} = \frac{2500}{3}$$

$$\mathrm{var}(W) = \frac{20^2}{12} = \frac{400}{12} = \frac{100}{3}$$

$$\mathrm{var}(Y) = \mathrm{var}(X) + \mathrm{var}(W) = \frac{2600}{3}$$ (Since X, W are indep)

$$\begin{aligned} \mathrm{Cov}(X,Y) &= \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[X(X+W)] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[X^2] + \mathbb{E}[X]\mathbb{E}[W] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathrm{var}(X) + (\mathbb{E}[X])^2 + \mathbb{E}[X]\mathbb{E}[W] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \frac{2500}{3} + 50^2 + (50)(10) - (50)(60) \\ &= \frac{2500}{3} + 2500 + 500 - 3000 \\ &= \frac{2500}{3} \end{aligned}$$

$$\begin{aligned} LLSE(X|Y=y) &= 50 + \left(\frac{2500}{3}\right)\left(\frac{3}{2600}\right)(y-60) \\ &= 50 + \frac{25}{26}(y-60) \end{aligned}$$

Extra page

(1.2.d cont)

(ii) $X \sim N(0,100)$, $W \sim N(0,20)$, $Y = X + W$ so $Y \sim N(\mathbb{E}[X] + \mathbb{E}[W], var(X) + var(W))$
$Y \sim N(0, 120)$

X and Y are Jointly Gaussian (we'll calculate the distribution to show this is true)

$$cov(X,Y) = \mathbb{E}[(X - \cancel{\mathbb{E}[X]}^{0})(Y - \cancel{\mathbb{E}[Y]}^{0})]$$
$$= \mathbb{E}[XY]$$
$$= \mathbb{E}[X(X+W)]$$
$$= \mathbb{E}[X^2] + \mathbb{E}[X]\cancel{\mathbb{E}[W]}^{0}$$
$$= 100.$$

Then we can write X and Y in terms of standard normal rvs U and V where $\binom{U}{V} \sim N(0, I)$ to get

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \sqrt{100} & 0 \\ \sqrt{100} & \sqrt{20} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$

so X and Y are Jointly Gaussian.

With Jointly Gaussian rv's,

MMSE = LLSE and

$$MMSE = \mathbb{E}[X \mid Y = y] = \mathbb{E}[X] + \Sigma_{XY} \Sigma_Y^{-1} (y - \mathbb{E}[Y])$$
$$= 0 + 100\left(\frac{1}{120}\right)(y - 0)$$
$$= \frac{5}{6} y$$

(1.2.d cont)

(iii) What is the mean squared error in each of the estimates of parts (i) and (ii)?

MMSE, unif:

$$\text{mean squared error} = \mathbb{E}[(X-\mathbb{E}[X|Y])^2]$$
$$= \mathbb{E}\left[\mathbb{E}[(X-\mathbb{E}[X|Y])^2|Y]\right]$$
$$= \mathbb{E}[\mathrm{var}(X|Y)]$$

For this case,

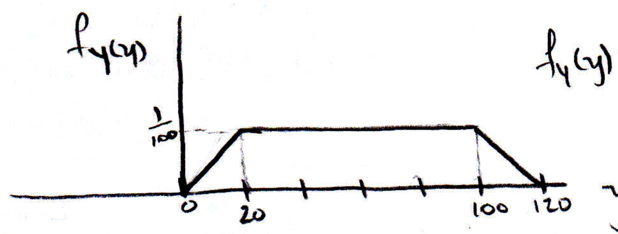
$\mathrm{var}(X|Y=y) = \mathrm{var}(\mathrm{Unif}(\min\{X|Y=y\}, \max\{X|Y=y\}))$ where the boundaries are determined by the picture a couple pages back.

$$= \begin{cases} y^2/12 & 0 \le y \le 20 \quad \text{since on this range } X\sim \mathrm{unif}[0,y] \\ \frac{400}{12} = \frac{100}{3} & 20 \le y \le 100 \\ \frac{(120-y)^2}{12} & 100 \le y \le 120 \end{cases}$$

So now we need

$$\mathbb{E}[\mathrm{var}(X|Y)] = \int_0^{20} y^2/12\, f_Y(y)\,dy + \int_{20}^{100} \tfrac{100}{3} f_Y(y)\,dy + \int_{100}^{120} \frac{(120-y)^2}{12} f_Y(y)\,dy$$

where



$$f_Y(y) = \begin{cases} \frac{y}{2000} & 0 \le y \le 20 \\ \frac{1}{100} & 20 \le y \le 100 \\ \frac{-y}{2000} + \frac{3}{50} & 100 \le y \le 120 \end{cases}$$

$$\mathbb{E}[\mathrm{var}(X|Y)] = \int_0^{2} \frac{y^3}{12\cdot 2000}\,dy + \int_{20}^{100} \frac{1}{3}\,dy + \int_{100}^{120} \frac{(120-y)^2(120-y)}{12\cdot 2000}\,dy$$

$$= \frac{1}{4}\frac{y^4}{12\cdot 2000}\Big|_0^2 + \frac{1}{3}y\Big|_{20}^{100} + \frac{-1}{4\cdot 12\cdot 2000}(y-120)^4\Big|_{100}^{120}$$

$$= \frac{1}{3\cdot 2000} + \frac{100}{3} - \frac{20}{3} + \frac{1}{4\cdot 12\cdot 2000}(20^4)$$

$$= 28.3335$$

(1.2.d.iii cont)

LLSE, unif

mean squared error $= \Sigma_X - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_{YX}$

$= \frac{2500}{3} - \left(\frac{2500}{3}\right)^2\left(\frac{3}{2600}\right)$ from pt (i)

$\simeq 32$

LLSE/MMSE, Gaussian

$$
\begin{aligned}
MSE &= var(X|Y) \\
&= \Sigma_X - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_{YX} \\
&= 100 - (100)^2\left(\frac{1}{120}\right) \\
&= \frac{\cancel{100(100)} + 100(20) - \cancel{100(100)}}{120} \\
&= \frac{50}{3}
\end{aligned}
$$

(iv) There is a 2nd lemming at Y_2. New estimate? Y_2 has same marginal as Y_1

We need

$$
\begin{aligned}
Cov(Y_2, Y_1) &= \mathbb{E}[(Y_2 - \mathbb{E}[Y_2])(Y_1 - \mathbb{E}[Y_1])] \\
&= \mathbb{E}[(X+W_2)(X+W_1)] \quad \text{where } W_1, W_2 \text{ are indep wanderings of lemmings 1 and 2.} \\
&= \mathbb{E}[X^2] + \cancel{\mathbb{E}[X]\mathbb{E}[W_1]}^{0} + \cancel{\mathbb{E}[X]\mathbb{E}[W_2]}^{0} + \cancel{\mathbb{E}[W_2]\mathbb{E}[W_1]}^{0} \\
&= 100
\end{aligned}
$$

Then

$$
\begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix} \sim N\left(0, \begin{pmatrix} \overset{\Sigma_X}{100} & \overset{\Sigma_{XY}}{100 \quad 100} \\ \begin{matrix}100 \\ 100\end{matrix}_{\Sigma_{YX}} & \begin{matrix}120 & 100 \\ 100 & 120\end{matrix}_{\Sigma_Y} \end{pmatrix}\right)
$$

and new estimate

$$
\begin{aligned}
\mathbb{E}[X|Y_1, Y_2] &= \mathbb{E}[X] + \Sigma_{XY}\Sigma_Y^{-1}\left(\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} - \begin{pmatrix} \mathbb{E}[Y_1] \\ \mathbb{E}[Y_2] \end{pmatrix}\right) \\
&= 0 + \begin{bmatrix} 100 & 100 \end{bmatrix}\begin{bmatrix} 120 & 100 \\ 100 & 120 \end{bmatrix}^{-1}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
&= \frac{1}{120^2 - 100^2}\begin{bmatrix} 100 & 100 \end{bmatrix}\begin{bmatrix} 120 & -100 \\ -100 & 120 \end{bmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
&= \frac{12000 - 10000}{120^2 - 100^2}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \frac{5}{11}(Y_1 + Y_2)
\end{aligned}
$$

e. 15 pts *You have one lemming starting at point 0. It walks in the positive direction until it randomly decides to turn around and then wanders in the negative direction until it again randomly decides to turn around and starts wandering in the positive direction again. This continues. In either direction, the lemming will walk an independent length $X_i \sim Exp(\lambda)$ before it decides to turn around.*

i. *After the lemming has turned around 2n times, use the CLT to approximate the probability that the lemming is more than d away from where it started.*

ii. *After the lemming has turned around 2n times, use the CLT to approximate the probability that the total distance the lemming has walked (what its pedometer would read at this point) is more than d.*

(i) Let Y_i = signed distance traveled during length i $\left(\text{so } Y_i = \begin{cases} X_i \text{ for } i = 2k+1, \; k = 0 \dots n-1 \\ -X_i \text{ for } i = 2k, \; k = 1 \dots n \end{cases} \right)$

Then, we want

$$P\left(\left|\sum_{i=1}^{2n} Y_i\right| > d\right) = P\left(\left|\sum_{i \text{ odd}} X_i - \sum_{i \text{ even}} X_i\right| > d\right)$$

Let's approximate Z_{odd} = CLT approx of $\sum_{i \text{ odd}} X_i$

and Z_{even} = CLT approx of $\sum_{i \text{ even}} X_i$

Then Z = CLT approx of $\sum_i Y_i$
$= Z_{odd} - Z_{even}$

and we then only need $P\left(\left|\sum_{i=1}^{2n} Y_i\right| > d\right) \simeq 2P(Z > d)$

For Z_{odd}: by CLT we have

$$P\left(\frac{\sum_{i \text{ odd}} X_i - n\mathbb{E}[X_i]}{\sqrt{n \operatorname{var}(X_i)}} < z\right) \simeq P(N(0,1) < z)$$

$$P\left(\sum_{i \text{ odd}} X_i < z\sqrt{n \operatorname{var}(X_i)} + n\mathbb{E}[X_i]\right) \simeq P\left(N\left(n\mathbb{E}[X_i], n\operatorname{var}(X_i)\right) < z\sqrt{n\operatorname{var}(X_i)} + n\mathbb{E}[X_i]\right)$$

$$\text{so } \sum_{i \text{ odd}} X_i \simeq_D N\left(n\mathbb{E}[X_i], n\operatorname{var}(X_i)\right)$$
$$\simeq_D N\left(\frac{n}{\lambda}, \frac{n}{\lambda^2}\right) \sim Z_{odd}.$$

(1.2.e.i cont)

Now, X_i's are $\mathrm{Exp}(\lambda)$ whether i is odd or even, and the approximation for i even also contains n elements to the sum.

$$\therefore \; Z_{even} =_D Z_{odd}$$

$$Z_{even} \sim N\left(\frac{n}{\lambda}, \frac{n}{\lambda^2}\right)$$

So $Z = Z_{odd} - Z_{even}$ which is also Gaussian since Z_{odd} and Z_{even} are independent Gaussians

$$\mathbb{E}[Z] = \mathbb{E}[Z_{odd}] - \mathbb{E}[Z_{even}] = 0$$

$$\mathrm{var}(Z) = \mathrm{var}(Z_{odd}) + \mathrm{var}(Z_{even}) = \frac{2n}{\lambda^2}$$

and $Z \sim N\left(0, \frac{2n}{\lambda^2}\right)$

So

$$P\left(\left|\sum_{i=1}^{2n} Y_i\right| > d\right) \simeq 2P(Z > d)$$

$$= 2P\left(N\left(0, \frac{2n}{\lambda^2}\right) > d\right)$$

$$= 2P\left(N(0,1) > \frac{d\lambda}{\sqrt{2n}}\right)$$

$$= 2P\left(N(0,1) < -\frac{d\lambda}{\sqrt{2n}}\right)$$

$$= 2\Phi\left(-\frac{d\lambda}{\sqrt{2n}}\right)$$

(ii) Use CLT to estimate the probability the total distance is $> d$.

Now we want

$$P\left(\sum_{i=1}^{2n} X_i > d\right) = P\left(\sum_{i\,odd} X_i + \sum_{i\,even} X_i > d\right) \approx P(Z_{ped} > d)$$

where $Z_{ped} = Z_{odd} + Z_{even}$

so $\mathbb{E}[Z_{ped}] = \mathbb{E}[Z_{odd}] + \mathbb{E}[Z_{even}] = \frac{2n}{\lambda}$

$\mathrm{var}(Z_{ped}) = \mathrm{var}(Z_{odd}) + \mathrm{var}(Z_{even}) = \frac{2n}{\lambda^2}$

$Z_{ped} \sim N\left(\frac{2n}{\lambda}, \frac{2n}{\lambda^2}\right)$

And

$$P\left(\sum_{i=1}^{2n} X_i > d\right) \simeq P(Z_{ped} > d)$$

$$= P\left(N\left(\frac{2n}{\lambda}, \frac{2n}{\lambda^2}\right) > d\right) = P\left(N(0,1) > \left(d - \frac{2n}{\lambda}\right)\frac{\lambda}{\sqrt{2n}}\right)$$

$$= P\left(N(0,1) < \left(\frac{2n}{\lambda} - d\right)\frac{\lambda}{\sqrt{2n}}\right)$$

$$= \Phi\left(\left(\frac{2n}{\lambda} - d\right)\frac{\lambda}{\sqrt{2n}}\right)$$