UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME132 Dynamic Systems and Feedback

Midterm I — Spring 2010

Closed Book and Closed Notes. One 8.5 × 11 sheet (only front) of handwritten notes allowed. Scientific calculator without graphics allowed.

Your Name:

Please answer all questions.

Problem:	1	2	3	4	Total
Max. Grade:	25	20	40	15	100
Grade:					

1. A LTI system is described by the following differential equations:

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) + x_2(t) + u(t) \\ \dot{x}_2(t) &= -2x_1(t) \\ y(t) &= x_2(t) \end{aligned}$$

where $u(t)$ is the input and $y(t)$ is the output. The initial conditions are $x_1(0) = x_{1,0}$ and $x_2(0) = x_{2,0}$.

(a) Write down the matrices A, B, C, and D of the state-space form.

The state-space form is

$$\begin{aligned} \dot{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} &= \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u. \end{aligned}$$

The matrices are: $A = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$, and $D = 0$.

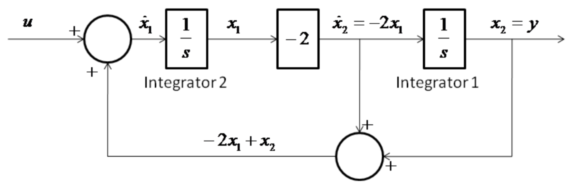
(b) What is the single, linear ODE (SLODE) relating the input $u(t)$ to the output $y(t)$?

Substitute $y = x_2$ in the second ODE ($\dot{x}_2 = -2x_1$), we obtain: $x_1 = \frac{-\dot{y}}{2}$. Take the time derivative of x_1, i.e. $\dot{x}_1 = \frac{-\ddot{y}}{2}$, and substitue in the first ODE ($\dot{x}_1 = -2x_1 + x_2 + u$) to obtain the SLODE

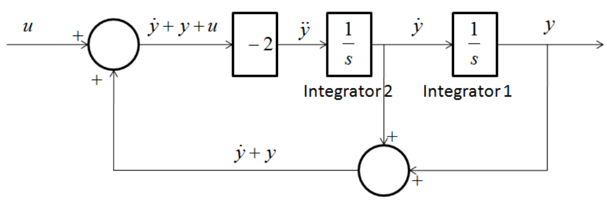
$$\ddot{y} = -2\dot{y} - 2y - 2u$$

with initial conditions: $\dot{y}(0) = -2x_{1,0}$, and $y(0) = x_{2,0}$.

(c) Sketch the Simulink block diagram for the LTI system by composing integrators. Show also where the initial conditions $x_{1,0}$ and $x_{2,0}$ are set.



“Integrator 1” is initialized to $x_{2,0}$, and “Integrator 2” is initialized to $x_{1,0}$.
OR



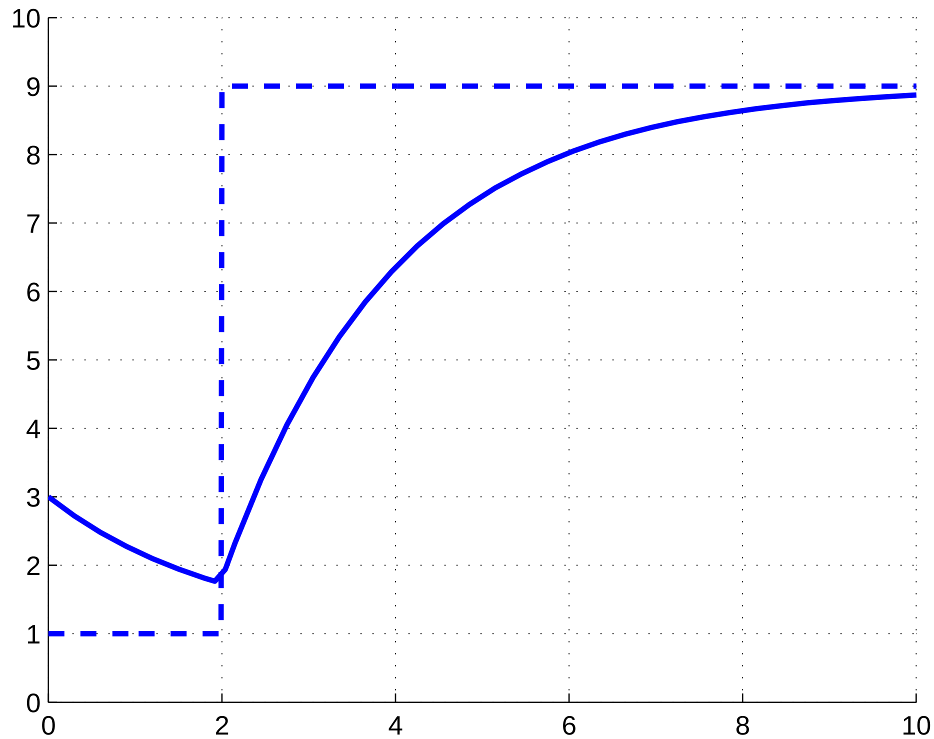
“Integrator 1” is initialized to $x_{2,0}$, and “Integrator 2” is initialized to $-2x_{1,0}$.

2. Consider the LTI system described by the differential equation

$$\dot{y} + \frac{1}{2}y = u + 2d$$

where $u(t)$ is the input, $d(t)$ is the disturbance, and $y(t)$ is the output. The initial condition is $y(0) = 3$.

Let $u(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t \geq 0 \end{cases}$, and $d(t) = \begin{cases} 0 & t < 2 \\ 2 & t \geq 2 \end{cases}$. Sketch the response $y(t)$ on the graph provided below.



3. Consider the LTI system described by the differential equation

$$\dot{y} + 4y = u \tag{1}$$

where $u(t)$ is the input, and $y(t)$ is the output. The initial condition is $y(0) = 0$. Assume that the sensor measuting $y(t)$ is affected by noise:

$$y_m(t) = y(t) + \eta(t)$$

where $y_m(t)$ is the sensor output, $\eta(t)$ is the noise. Consider the feedback controller

$$u(t) = K(r(t) - y_m(t))$$

where $r(t)$ is a reference signal.

(a) Is the system described by equation (1) stable?
The system is stable ($4 > 0$).

(b) Write the closed-loop differential equation.
The closed-loop differential equation is obtained by substituting $u = K(r - (y + \eta))$ into equation (1) :

$$\dot{y} + (K + 4)y = K(r - \eta).$$

(c) Which values of the controller K guarantee closed-loop stability?
Stability is guaranteed when $(K + 4) > 0$ or $K > -4$.

(d) Assume $r(t) = 0$. Design a controller K such that (1) the closed-loop system is stable and (2) the magnitude of the output at steady-state is at most 0.1 when the sensor measurement $y_m(t)$ is affected with a constant noise $\eta(t)$ of magnitude 0.2 . Report the value of the controller K you choose to use.

Hint: Note that condition (2) means that we want 50% noise reduction. Rewrite condition (2) as $|y_{ss}| \leq 0.1$ when $\eta(t) = 0.2$ for all $t \geq 0$ and $r(t) = 0$

For $r(t) = 0$ and $\eta(t) =$ constant, the steady-state output is:

$$y_{ss} = \frac{-K\eta}{K+4}.$$

Condition (2), i.e. $|y_{ss}| \leq 0.1$, is satisfied when $\left|\frac{-K(0.2)}{K+4}\right| \leq 0.1$ or $-\frac{4}{3} \leq K \leq 4$.

(a) List advantages and disadvantages in using a controller K much bigger that the one you selected in (d).

Advantages

i. Faster Tracking (time constant $T = \frac{1}{K+4}$ is reduced)

ii. Improved steady-state disturbance rejection (If there is disturbance, the closed-loop differential equation is $\dot{y} + (K+4)y = K(r - \eta) + d$. The term $\left(\frac{d}{K+4}\right)$ will get smaller for larger value of K.

Disadvantages

i. Larger values of K might lead to actuator saturation, or damage of the physical system.

(b) Consider the controller you designed in (d). What is the maximum output that the system achieves when $r = 1$ and noise magnitude is bounded by 0.2. (i.e. $|\eta(t)| \leq 0.2$).

I will choose $K = 3$, for $r = 1$ we have $\dot{y} + 7y = 3(1 - \eta)$. The maximum output at steady-state is achieved when $\eta = -0.2$:

$$y_{ss} = \frac{3}{7}(1 - (-0.2)) = 0.51.$$

4. Consider the LTI system described by the differential equation

$$\dot{y} + 2y = u$$

where $u(t) = \begin{cases} 0 & t \leq 0 \\ sin(\omega t) & t > 0 \end{cases}$ is the input, and $y(t)$ is the output.

(a) Write the steady-state output $y_{ss}(t)$ when $\omega = \pi$?

$$y_{ss} = M(\pi)sin(\pi t + \phi),$$

where $M(\pi) = \frac{1}{\sqrt{4+\pi^2}} \approx 0.3$, and $\phi = tan^{-1}(-\frac{\pi}{2}) \approx -1$. (i.e. $y_{ss} \approx 0.3sin(\pi(t - 0.32)))$.

(b) If we double the the frequency of the input (*i.e.* $\omega = 2\pi$). What happens to the magnitude of the steady-state output? (Does it increase/decrease? By how much?). What about the phase shift?

If $\omega = 2\pi$, then $y_{ss} \approx 0.15sin(2\pi(t - 0.16))$. The magnitude of the steady-state output decreases by a factor of ≈ 0.5, and the output shifts to the left by $t \approx 0.16\, sec$ as shown in the figure below.