Solutions to the Second Midterm Exam – Multivariable Calculus

Math 53, November 5, 2009. Instructor: E. Frenkel

1. The height of a mountain above the point (x, y) is given by the formula $h(x, y) = x^2 + \sin^2(xy)$. A climber is standing on the mountain at the point $(1, 0, 1)$.

(a) In the direction of what unit vector should the climber move (with respect to the x, y coordinate plane) so as to achieve the steepest ascent?

We find: $h_x = 2x + 2\sin(xy)\cos(xy)y, h_y = 2\sin(xy)\cos(xy)x$. Hence the gradient at the point $(1, 0)$ is $\vec{\nabla}h(1, 0) = \langle 2, 0\rangle$. The steepest ascent rate is achieved in the direction of the unit vector $\langle 1, 0\rangle$.

(b) In the direction of what unit vectors should the climber move to achieve the ascent rate equal to 50% of the steepest ascent rate?

The ascent rate in the direction of a unit vector $\mathbf{u}$ equals $\mathbf{u}\cdot\vec{\nabla}h(1, 0) = |\vec{\nabla}h(1, 0)|\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\vec{\nabla}h(1, 0)$.

In order to achieve the ascent rate equal to 50% of the steepest ascent rate the climber should move in a direction that has the angle θ such that $\cos\theta = 1/2$. Therefore $\theta = \pm\pi/3$, and we get $\mathbf{u} = \langle 1/2, \pm\sqrt{3}/2\rangle$.

2. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - y^2 + x^2y + 4$ on the set $D = \{(x, y) \mid |x| \le 1, |y| \le 1\}$. Find the points at which these values are attained.

First, we find the critical points on the interior of this set. Setting $f_x = 2x+2xy = 0$, $f_y = -2y + x^2 = 0$, we obtain one solution: $x = 0, y = 0$, which is in D (otherwise, $x^2 = -2$, which cannot be solved).

Next, we restrict our function to each segment of the boundary: (1) $x = 1$: $f \mapsto -y^2 + y + 5$. Its derivative is $-2y + 1$, and hence $y = 1/2$. (2) $x = -1$: obtain the same function, so again $y = 1/2$. (3) $y = 1$: $f \mapsto 2x^2 + 5$. Its derivative is $4x$, and hence $x = 0$. (4) $y = -1$: $f \mapsto 3$, and the derivative is identically equal to 0. So we have to consider all points of this segment.

Including the corners, we assemble the following list: $(0, 0), (1, 1/2), (-1, 1/2), (0, 1)$, $\{(x, -1) \mid |x| \le 1\}$, $(1, 1), (-1, 1)$. Evaluating the function at all of these points, we find that the maximum value $5\frac{1}{4}$ is attained at the points $(1, 1/2)$ and $(-1, 1/2)$, and the minimal value 3 is attained at $(0, 1)$ and on the segment $\{(x, -1) \mid |x| \le 1\}$.

3. Find the maximum and minimum values of the function $f(x, y, z) = xyz$ subject to the constraint $x^2 + 2y^2 + 3z^2 = 6$. At how many points is each of these values attained? List all of these points.

We use Lagrange method. The system of equations $\vec{\nabla}f = \lambda\vec{\nabla}g, g = k$ reads $yz = 2\lambda x, xz = 4\lambda y, xy = 6\lambda z, x^2 + 2y^2 + 3z^2 = 6$. The first three equations imply that $x^2 = xyz/2\lambda, y^2 = xyz/4\lambda, z^2 = xyz/6\lambda$. Hence $y^2 = x^2/2, z^2 = x^2/3$. Substituting in the last equation, we find that $3x^2 = 6$. Hence $x = \pm\sqrt{2}$, and so $y = \pm 1$, $z = \pm\sqrt{2}/\sqrt{3}$. But the signs can be chosen independently! The maximum value is thus $2/\sqrt{3}$, achieved at the points of the form $(\pm\sqrt{2}, \pm 1, \pm\sqrt{2}/\sqrt{3})$, with the number of $-$ signs even (there are 4 such points), and at the minimum value if $-2/\sqrt{3}$, achieved at the points of the same form with with the number of $-$ signs odd (there are 4 of those).

4. Evaluate the integral

$$\int_0^1 \int_{\sqrt[3]{y}}^1 \sqrt{x^4+1}\, dx\, dy.$$

The region of integration is

$$\{(x,y) \mid 0 \le y \le 1, \sqrt[3]{y} \le x \le 1\} = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^3\}.$$

Changing the order of integration, obtain the integral

$$\int_0^1 \int_0^{x^3} \sqrt{x^4+1}\, dydx = \int_0^1 x^3\sqrt{x^4+1}\, dx = \frac{1}{6}(x^4+1)^{3/2}\Big|_0^1 = \frac{1}{6}(2\sqrt{2}-1).$$

5. A lamina occupies the region in the first quadrant on the xy plane bounded by the ellipse $25x^2+4y^2=1$. Its density of mass function is given by the formula $\cos(25x^2 + 4y^2)$. Find the mass of the lamina.

We need to evaluate the following integral:

$$\int\int_R \cos(25x^2+4y^2) dA.$$

Make the change of variables: $x = \frac{1}{5} r\cos\theta, y = \frac{1}{2} r\sin\theta$. The absolute value of the Jacobian for this change equals $\frac{r}{10}$ (since $r \ge 0$). Hence in the r,θ–coordinates the integral becomes:

$$\frac{1}{10}\int_0^{\pi/2}\int_0^1 \cos(r^2) r dr d\theta = \frac{\pi}{40}\sin(1).$$

6. Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=2$ and is confined between the yz plane and the cone $x = \sqrt{y^2+z^2}$.

Observe that the volume of this solid is equal to the volume of the solid E that lies within the sphere $x^2+y^2+z^2=2$ and is confined between the xy plane and the cone $z=\sqrt{x^2+y^2}$. The latter is equal to the triple integral of the function 1 over E. To evaluate it, we use spherical coordinates. We obtain

$$\int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^{\sqrt{2}} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{1}{3} 2\sqrt{2}\cdot 2\pi \cdot \frac{1}{\sqrt{2}} = \frac{4\pi}{3}.$$