MATH 110

First Midterm Examination
February 16, 2010
2:10–3:30 PM, 10 Evans Hall

Please put away all books, calculators, and other portable electronic devices—anything with an ON/OFF switch. You may refer to a single 2-sided sheet of notes. For numerical questions, *show your work* but do not worry about simplifying answers. For proofs, write your arguments in complete sentences that explain what you are doing. Remember that your paper becomes your only representative after the exam is over.

Problem	Your score	Possible points
1		5 points
2		12 points
3		6 points
4		7 points
Total:		30 points

1. In $\mathbf{R}^3$, express $(3, 18, -11)$ as a linear combination of $(1, 2, 3)$, $(-2, 0, 3)$ and $(2, 4, 1)$.

This was a standard numerical problem of the type that most of you know how to do. The coefficients are: $-49/5$, 3 and $47/5$. I apologize for the fractions: I intended the answers to be whole numbers and must have mistyped.

2. Label each of the following statements as TRUE or FALSE. Along with your answer, provide an informal proof or an explanation. For false statements, an explicit counterexample might work best. In interpreting the statements, take v to be a vector, a to be a scalar, β to be a basis of V, etc., etc.

Each part was worth 2 points. We gave out one point for the correct T/F answer and one point for the explanation.

a. If $av = v$, then either $a = 1$ or $v = 0$.

This is true, but a lot of you didn't give a good reason. Since $v = 1 \cdot v$, as proved in class, the equation $av = v$ may be written $(a - 1) \cdot v = 0$. If the scalar $a - 1$ is non-zero, we may divide by it (i.e., multiply by its inverse) and get $v = 0$. In other words, if a isn't 1, v is 0. This means that we have $a = 1$ or $v = 0$, or both.

b. If A and B are real 3×3 matrices, the formula $T(M) = AM - MB$ defines a linear map $\mathsf{M}_{3\times3}(\mathbf{R}) \to \mathsf{M}_{3\times3}(\mathbf{R})$.

This is true because of the distribution relations for matrix multiplication. For example, to see that $T(M + M') = T(M) + T(M')$, we have to expand out $A(M + M') + (M + M')B$ and rearrange terms.

c. If V is spanned by a set of 6 distinct vectors, all bases of V have exactly 6 vectors.

This is false. For example, the 1-dimensional $\mathbf{R}$-vector space $\mathbf{R} = \mathbf{R}^1$ is spanned by the 6 distinct elements 1, 2, 3, 4, 5 and 6, but all bases of this space have one element!

d. If W is a subspace of a finite-dimensional vector space V and $w_1, \ldots, w_m$ form an ordered basis of W, then every basis of V includes $w_1, \ldots, w_m$.

This, again, is silly. Take W to be the subspace of $\mathbf{R}^2$ generated by $(1, 0)$, so that $(1, 0)$ is a basis of W. You can find lots of bases of $\mathbf{R}^2$ that do not contain $(1, 0)$. One such basis consists of $(1, 1)$ and $(0, 1)$.

e. In $\mathcal{L}(F^6, F^4)$, one may find linear transformations T for which the dimensions of $\mathsf{N}(T)$ are 2, 3, 4, 5 and 6.

This is true. Just make up 6×4 matrices of 0s and 1s with exactly i linearly independent columns, where i takes each of the values 0, 1, 2, 3 and 4.

f. If $m = \dim(V)$ and $n = \dim(W)$, then $[T]_\beta^\gamma$ is an $n \times m$ matrix. (Here T is a linear map $V \to W$.)

Well, this is just true, by definition of $[T]_\beta^\gamma$. There was a nearly identical question in the homework, but the HW answer was "false." I exchanged m and n and made the statement true instead!

3. Suppose that V is an F-vector space with at least three vectors. Let w be a vector in V. Prove that V is spanned by the set $S = \{ v \in V \mid v \neq w \}$.

This caused a lot of trouble, sorry. The span of S certainly contains all vectors in S. There's only one vector in V that isn't in S, namely w. Therefore, to prove that the span of S is all of V, we just have to prove that w is in the span! For every v in the vector space, we have

$$w = (w - v) + v.$$

This will write w in the span of S provided that the two summands v and $w - v$ are in S. To have v in S, we need to have $v \neq w$. To have $w - v \in S$, we need to have $v \neq 0$. Since V has more than two vectors, there is a $v \in V$ different from both 0 and w. Take such a v and we're home.

4. Let $f(x)$ be a polynomial of degree n with real coefficients. Prove that the $n+1$ polynomials

$$f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$$

are linearly independent. Conclude that they span $\mathsf{P}_n(\mathbf{R})$.

For the first statement, there are several correct proofs. One way to proceed is to realize that the last (i.e., nth) derivative is a non-zero constant because $f(x)$ has degree n. Hence the set consisting of the last vector $f^{(n)}(x)$ is linearly independent. The vector $f^{(n-1)}(x)$ then has degree 1, so it can't be a multiple of the vector $f^{(n)}(x)$. Thus the two vectors $f^{(n)}(x)$ and $f^{(n-1)}(x)$ form a linearly independent set. We proceed in this manner, incrementing the number of vectors in the set that we are proving to be linearly independent. At one stage we have seen that $f^{(n)}(x), f^{(n-1)}(x), \ldots, f^{(n-i)}(x)$ make a linearly independent set and ask whether the larger set $f^{(n)}(x), f^{(n-1)}(x), \ldots, f^{(n-i)}(x), f^{(n-i-1)}(x)$ is also linearly independent. If not, then $f^{(n-i-1)}(x)$ will be a linear combination of $f^{(n)}(x), f^{(n-1)}(x), \ldots, f^{(n-i)}(x)$. You can see that this is impossible because the degree of $f^{(n-i-1)}(x)$ is larger than the degrees of the polynomials $f^{(n)}(x), f^{(n-1)}(x), \ldots, f^{(n-i)}(x)$.

Maybe a better way to proceed is to start with

$$a_0 f(x) + \cdots + a_n f^{(n)}(x) = 0 \tag{1}$$

and to prove in turn that each of the a_i is 0. Note that all derivatives of $f(x)$ after the nth derivative are 0. If we differentiate (1) n times, all terms but the first disappear; we get $a_0 f^{(n)}(x) = 0$. Since the nth derivative is non-zero (it's in fact a non-zero constant), we get $a_0 = 0$. Hence the first term in (1) is really $a_1 f'(x)$. Now differentiate (1) $n-1$ times instead of n times; we get $a_1 f^{(n)}(x) = 0$, so $a_1 = 0$. We continue in this fashion, knocking off the terms one by one. At the end of the game, (1) has only one term left: it reads $a_n f^{(n)}(x) = 0$. We get $a_n = 0$, so all coefficients are 0, as was required.

For the second statement, which was worth two points, you just have to say that the polynomials $f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$ are known to be linearly independent by the first part. There are $n+1$ of them, and $n+1$ is the dimension of $\mathsf{P}_n(\mathbf{R})$. By a corollary to the theorem that many of you knew the number of, the linearly independent set

$$\{\, f(x), f'(x), f''(x), \ldots, f^{(n)}(x) \,\}$$

is actually a basis of $\mathsf{P}_n(\mathbf{R})$.