MATH 1B

Lec. 3, Spring 2010
Midterm 2
Solutions and Grading Rubric
Final Version

March 22, 2010

Preceding the solution is the number of points given for the full question. Preceding each main step of the solution is the number of points for that step.

1

Let
$$a_n = (-1)^n \frac{2n^2 + 3}{n^2 + n + 1}.$$

a) Determine whether the sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges. If it converges, find what it converges to. If it does not converge, state the reason why.

b) Determine whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges. If it converges, find what it converges to. If it does not converge, state the reason why.

Solution, total 4 points:

(2 points) Part a)

a) The sequence diverges, because the even terms converge to 2 and the odd terms converge to -2.

When n is even, $(-1)^n = 1$, and when n is odd $(-1)^n = -1$.

Therefore, for even n,

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2n^2 + 3}{n^2 + n + 1} = \lim_{x \to \infty} \frac{2x^2 + 3}{x^2 + x + 1} = (L'H) \lim_{x \to \infty} \frac{4x}{2x + 1} = (L'H) \lim_{x \to \infty} \frac{4}{2} = 2$$

However, for odd n,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} -\frac{2n^2+3}{n^2+n+1} = \lim_{x\to\infty} -\frac{2x^2+3}{x^2+x+1} = (L'H)\lim_{x\to\infty} -\frac{4x}{2x+1} = (L'H)\lim_{x\to\infty} -\frac{4}{2} = -2$$

(2 points) Part b)

b) The series diverges: The terms do not converge to 0.

In order for the series $\sum_{n=1}^{\infty} a_n$ to converge, it is necessary that the terms converge to 0, i.e., that $\lim_{n\to\infty} a_n = 0$. In part a) I show that $\lim_{n\to\infty} a_n$ does not exist, so it cannot be 0, and therefore $\sum_{n=1}^{\infty} a_n$ diverges.

2

Find the sum $\sum_{n=1}^{\infty} a_n$, where

$$a_n = \frac{1}{n^3} - \frac{1}{(n+1)^3}.$$

by computing the partial sums and taking a limit

Solution, 4 points total

This is a telescoping series, with partial sums

(1 point) for the first few terms

$$s_1 = a_1 = \frac{1}{1^3} - \frac{1}{2^3}$$

$$s_2 = a_1 + a_2 = \frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{2^3} - \frac{1}{3^3} = 1 - \frac{1}{3^3}$$

(1 point) for the general term

$$s_n = 1 - \frac{1}{(n+1)^3}$$

(2 points) Finding the Limit: 1 point for writing the series as a limit of partial sums, 1 point for evaluation

2

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} 1 - \frac{1}{(n+1)^3} = 1$$

3

Use the integral test to determine whether

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n},$$

is convergent, or not. (Note that the series meets the conditions for the integral test.)

Solution, 4 points total

(1 point) Setup

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n},$$

converges if and only if $\int_2^{\infty} \frac{1}{x \ln x} dx$ does.

Let $u = lnx$ then $du = \frac{1}{x}dx$ and $\int \frac{1}{x \ln x} dx = \int \frac{1}{u} du = \ln u +$ constant.

(2 points) Evaluation of the integral

Whether through changing the limits of integration to those of u or by solving the indefinite integral using u and then reverting to x and its limits of integration, you will get

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \int_{ln2}^{\infty} \frac{1}{u} du = \lim_{t \to \infty} (\ln(\ln t) - \ln(\ln 2)) = \infty$$

since $\ln(\ln 2)$ is finite.

(1 point) Concluding the series is divergent

4

Determine if the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{4^n}.$$

is absolutely convergent, conditionally convergent or divergent.

Solution, 5 points total

The series converges absolutely. This can be shown in different ways; here I use the ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2\frac{4^n}{4^{n+1}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2\lim_{n\to\infty}\frac{1}{4} = \frac{1}{4} < 1$$

(2 points) Setting up the applicable tests: ratio, alternating (the most work), root, etc.

(2 point) Finding the limit

(1 point) Determining "absolutely convergent"

5

Determine if the series

$$\sum_{n=2}^{\infty}\left(\frac{n}{\ln n}\right)^n$$

is convergent or divergent.

Solution, 4 points total (breakdown at the discretion of the grader. Suggested

breakdown for the solution shown here: 2 points for the setup of the root test, 2 points for the proper evaluation)

This series diverges. There are different ways to show this, among them examination using the root (or possibly the ratio) test and showing the terms do not go to zero. This is a proof using the root test.

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[n]{(\frac{n}{lnn})^n} = \lim_{n \to \infty} \frac{n}{lnn} = \lim_{x \to \infty} \frac{x}{lnx} = (L'H) \lim_{x \to \infty} \frac{1}{\frac{1}{x}} = \infty.$$

6

Find the radius of convergence and the interval of convergence of the power series
$$\sum_{n=0}^{\infty} nx^n.$$

Which function of x does this sum up to? (Recall that the series is related in a simple way to another series you know well).

Solution, total 5 points

(2 points) Finding the radius of convergence: 1 pt for using a correct convergence test, 1 pt for a conclusion

Using the ratio test and the root test one can show that the radius is 1.

Using the ratio test: Let a_n be the nth term of the series,

$$\lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| = \lim_{n \to \infty} |\frac{(n+1)x^{n+1}}{nx^n}| = \lim_{n \to \infty} \frac{n+1}{n}|x| = \lim_{n \to \infty} \frac{n+1}{n} \lim_{n \to \infty} |x|$$

since both limits exist.

To make the ratio smaller than 1 and therefore attain convergence, we need x with $|x| < 1$.

(1 point) Behavior on the boundary

When $x = 1$, we have $\sum_{n=0}^{\infty} n$, and when $x = -1$, we get $\sum_{n=0}^{\infty} (-1)^n n$. Both diverge since, e.g., the terms do not go to 0.
The interval of convergence is (-1, 1).

(2 points) Finding the function: 1 point for noticing how it is related to $1/(1 - x)$, and 1 point for deducing a sum.

The series sums to the function $\frac{x}{(1-x)^2}$.

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

Differentiating both sides and using the rules of power series gives $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$

Multiplying both sides by x gives

$$\sum_{n=1}^{\infty} n x^n = \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

7

Find Taylor/MacLaurin series expansion for

$$f(x) = \exp(2x),$$

around $x = 0$, using the definition of the Taylor/MacLaurin series. Can you check your answer against something you already know?

Solution, 4 points total

(1 point) General term

The nth term of the Taylor series expanded at a is $\frac{f^{(n)}(a)}{n!}x^n$ and when expanded at 0 will be $\frac{f^{(n)}(0)}{n!}x^n$

(2 points) Finding the coefficients/terms when $f(x) = e^{2x}$,

$$f(0) = 1$$
$$f'(x) = 2e^{2x}, f'(0) = 2e^0 = 2$$
$$f''(x) = 4e^{2x}, f''(0) = 4e^0 = 4$$
$$f^{(3)}(x) = 2^3 e^{2x}, f^{(3)}(0) = 2^3 e^0 = 2^3$$

Therefore the nth coefficient is $\frac{2^n}{n!}$

(1 point) Final expansion

The Taylor series expansion of $f(x)$ is $\sum_0^\infty \frac{2^n x^n}{n!} = \sum_0^\infty \frac{(2x)^n}{n!}$.

Sidenote:
This expression may also be achieved by plugging $2x$ into the expansion for e^x.