1) $\int t^3 (\sin(t^4))^2 \, dt$

Let $u = t^4$ then $du = 4t^3 dt$

$\Rightarrow t^3 dt = \frac{1}{4} du$

so the integral becomes

$\frac{1}{4} \int (\sin(u))^2 du$ 1

$= \frac{1}{4} \cdot \frac{1}{2} \int (1 - \cos(2u)) du$ (double angle formula) 2

$= \frac{1}{8} \left(u - \frac{1}{2} \sin(2u) \right) + C$ 3

$= \boxed{\frac{1}{8} \left(t^4 - \frac{1}{2} \sin(2t^4) \right) + C}$ 4

2. $\int x(\sec x)^2\,dx$

Integrate by parts:

Let $u = x \quad dv = (\sec x)^2$

then $du = dx \quad v = \tan x$

Thus

$$\int x(\sec x)^2\,dx = x\tan x - \int \tan x\,dx$$ 1

Now $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Let $w = \cos x$

then $dw = -\sin x\,dx$

logarithm law!

$$\text{so } \int \frac{\sin x}{\cos x}\,dx = -\int \frac{dw}{w} = -\ln(|w|) + C = \ln(1/|w|) + C$$ 2

$= \ln(|\sec x|) + C$

Plugging this into ⊛, we get

$$\int x(\sec x)^2\,dx = \boxed{x\tan x - \ln(|\sec x|) + C}$$

(you could rewrite $x\tan x + \ln(|\cos x|) + C$ as well.

3

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3. $\displaystyle\int_0^{\frac{a\sqrt{2}}{2}} \frac{x^2\,dx}{(a^2-x^2)^{1/2}}$

Let $x = a\sin\theta \qquad -\pi/2 \le \theta \le \pi/2$ 1

Then $dx = a\cos\theta\, d\theta$

and $(a^2-x^2)^{1/2} = (a^2(1-\sin^2\theta))^{1/2}$

$= (a^2\cos^2\theta)^{1/2}$

$= a\cos\theta$

Limits of integration: x goes from 0 to $a\sqrt{2}/2$

$a\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$a\sin\theta = a\sqrt{2}/2 \Rightarrow \sin\theta = \sqrt{2}/2 \Rightarrow \theta = \pi/4$ 2

Putting it all together, the integral becomes

$$\int_0^{\pi/4} \frac{a^2\sin^2\theta \cdot a\cos\theta\, d\theta}{a\cos\theta}$$ 3

$= \displaystyle\int_0^{\pi/4} a^2\sin^2\theta\, d\theta = a^2\int_0^{\pi/4}\sin^2\theta\, d\theta$

$= a^2\cdot\frac{1}{2}\displaystyle\int_0^{\pi/4}(1-\cos(2\theta))\,d\theta$ (double angle formula)

$= \dfrac{a^2}{2}\Big|_0^{\pi/4}\left(\theta - \frac{1}{2}\sin(2\theta)\right)$ 4

$= a^2/2\left(\left(\pi/4 - \frac{1}{2}\sin(\pi/2)\right) - \left(0 - \cancel{\tfrac{1}{2}\sin(0)}^{\,0}\right)\right)$

$= a^2/2\,(\pi/4 - 1/2) = \boxed{a^2(\pi-2)/8}$

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4. $\int_0^1 \frac{x\arctan(x)}{(1+x^2)^{3/2}}\,dx$

Let $x = \tan y$ $\quad -\pi/2 < y < \pi/2$

Then $dx = \sec^2 y\, dy$

$\arctan(x) = \arctan(\tan(y)) = y$

$(1+x^2)^{3/2} = (1+\tan^2 y)^{3/2}$

$= (\sec^2 y)^{3/2}$

$= \sec^3 y$

1

Limits of integration: x goes from 0 to 1

When $\tan y = 0$, $y = 0$ ← because $\tan(0) = 0$

When $\tan y = 1$, $y = \pi/4$ $\quad \tan(\pi/4) = 1$

2

Putting all this together, the integral becomes

$\int_0^{\pi/4} (\tan y \cdot y \cdot \sec^2 y\, dy)/(\sec^3 y)$

$= \int_0^{\pi/4} (y \cdot \tan y\, dy)/\sec y = \int_0^{\pi/4} \left(y \cdot \frac{\sin y}{\cos y}\, dy\right)/(1/\cos y)$

3

$= \int_0^{\pi/4} y \sin y\, dy.$

Integrate by parts:

Let $u = y$ $\quad dv = \sin y\, dy \Rightarrow du = dy \quad v = -\cos y$

Then

$\int_0^{\pi/4} y \sin y\, dy = \Big|_0^{\pi/4} -y\cdot\cos y + \int_0^{\pi/4} \cos y\, dy$

$= \Big|_0^{\pi/4} (-y\cos y + \sin y)$

$= \left(-\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\right) - (-0\cdot\cos 0 + \ldots$

$= \boxed{\frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right)}$

5. $\displaystyle\int_1^\infty \frac{\ln x \, dx}{x(2\ln x+1)(\ln x+1)}$

First, note that the integrand is discontinuous only when $x=0$, when $2\ln x + 1 = 0 \Rightarrow \ln x = -1/2 \Rightarrow x = 1/\sqrt{e}$ and when $\ln x + 1 = 0 \Rightarrow \ln x = -1 \Rightarrow x = 1/e$.
None of these are in the domain of integration so this integral is only improper because the upper limit is ∞. 1

Let's evaluate this guy:

Let $u = \ln x$ then $du = \frac{1}{x}dx$

For the limits of integration: when $x=1$, $u = \ln(1) = 0$
as $x \to \infty$, $\ln(x) \to \infty$ also

So our integral becomes

$$\int_0^\infty \frac{u\,du}{(2u+1)(u+1)} = \lim_{t\to\infty} \int_0^t \frac{u\,du}{(2u+1)(u+1)}$$ 2

Partial fractions: $\dfrac{u}{(2u+1)(u+1)} = \dfrac{A}{2u+1} + \dfrac{B}{u+1}$

$$\Rightarrow u = A(u+1) + B(2u+1) = (A+2B)u + (A+B)$$

so
$$\begin{array}{r} A+2B=1 \\ -\ \ A+B=0 \\ \hline \boxed{B=1} \end{array} \qquad A+1=0 \quad \boxed{A=-1} \qquad \text{so } \frac{u}{(2u+1)(u+1)} = \frac{1}{u+1} - \frac{1}{2u+1}$$ 3

$$\text{So } \lim_{t\to\infty}\int_0^t \frac{u\,du}{(2u+1)(u+1)} = \lim_{t\to\infty}\int_0^t \left(\frac{1}{u+1} - \frac{1}{2u+1}\right)du$$

$$= \lim_{t\to\infty} \Big|_0^t \left[\ln|u+1| - \tfrac{1}{2}\ln\left|u+\tfrac{1}{2}\right|\right] = \lim_{t\to\infty}\Big|_0^t \ln\left|\frac{u+1}{\sqrt{u+\frac{1}{2}}}\right|$$

$$= \lim_{t\to\infty}\left(\ln\left|\frac{t+1}{\sqrt{t+1/2}}\right| - \ln\sqrt{2}\right) = \infty$$ so the integral diverges. 4

If we had integrated from $1/2$ to ∞ we would have needed to split up the integral into $\int_{1/2}^{1/\sqrt{e}} + \int_{1/\sqrt{e}}^{1} + \int_1^\infty$. 5

6) Show that

$$\int_1^\infty dx/(x+e^x)$$

converges or diverges using the Comparison Test.

We will compare the integrand to $1/e^x$. 1

Observe that $\displaystyle\int_1^\infty \frac{dx}{e^x} = \lim_{t\to\infty} \int_1^t \frac{dx}{e^x}$

$$= \lim_{t\to\infty} \Big|_1^t -e^{-x} = \lim_{t\to\infty} \left[-e^{-t} + e^{-1}\right] = e^{-1}$$

so $\displaystyle\int_1^\infty \frac{dx}{e^x}$ is convergent. 3

Furthermore, we have $\displaystyle 0 < \frac{1}{x+e^x} < \frac{1}{e^x}$

(since $x+e^x > e^x$) for all $x \geq 1$ (our domain of integration).

Therefore, $\displaystyle\int_1^\infty dx/(x+e^x)$

converges by the Comparison Theorem. 4

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7) Find the length of the curve given by

$$y = \frac{2}{3}x^{3/2}$$

from $x=0$ to $x=1$.

The length of $y=f(x)$ for $a \le x \le b$, provided $f'(x)$ is continuous, is

$$\int_a^b \sqrt{1+(f'(x))^2}\,dx$$ 1

Here $f(x) = \frac{2}{3}x^{3/2}$

$\Rightarrow f'(x) = \frac{2}{3}\cdot\frac{3}{2}\cdot x^{1/2} = x^{1/2}$

$\Rightarrow (f'(x))^2 = (x^{1/2})^2 = x$

so the length of our curve is

$$\int_0^1 \sqrt{1+x}\,dx$$ 2

Let $w = 1+x$ then $dw = dx$

Limits of integration: when $x=0$, $w = 1+0 = 1$

when $x=1$, $w = 1+1 = 2$

so our integral becomes

$$\int_1^2 \sqrt{w}\,dw = \int_1^2 w^{1/2}\,dw = \Big|_1^2 \frac{w^{3/2}}{3/2}$$ 3

$$= \frac{2}{3}\Big|_1^2 w^{3/2} = \frac{2}{3}\left(2^{3/2} - 1^{3/2}\right) = \boxed{\frac{2}{3}\left(2\sqrt{2}-1\right)}$$

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