Name:____

Math 54 Final Exam

August 14, 2009

Instructor: James Tener

Problem 1: ______ / 20 points

Problem 2: ______ / 10 points

Problem 3: ______ / 15 points

Problem 4: ______ / 10 points

Problem 5: ______ / 10 points

Problem 6: ______ / 15 points

Problem 7: ______ / 5 points

Total: ______ / 85 points

Instructions:

- Show all of your work. When justifying answers, express yourself clearly and in an organized fashion. You are graded on what you write down, not what you mean to say.
- You may cite theorems from class/the book by (correctly) stating what it says.
- Cross out any work you do not want graded.
- No calculators are allowed.

Problem 1. You do not need to justify your answers on this problem. (5 points each)

(a) Give an example of a matrix A and a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has a non-unique solution.

(b) Define what is meant by a <u>fundamental matrix</u> for the system of ODEs $\vec{x}\,' = A\vec{x}$. (Be sure to define it, not explain how to find it)

(c) Give an example of an infinite dimensional vector space.

(d) Let A be an $m \times n$ matrix, and let $\cdot$ be the dot product. Suppose $x = (Ay) \cdot z$ (note: vector arrows have been omitted). For each of x, y and z, say whether it is a vector or a scalar, and say how many entries each vector has.

Problem 2. Find all solutions to the ODE $y'' - 4y' + 5y = te^{2t}$ such that $y(0) = y'(0)$. (10 points)

Problem 3. Say whether the given statement is true or false. If it is true, explain why. If it is false, provide a counterexample showing that it is false. No points are given for true/false without correct justification (i.e. **no points for "false" without a concrete counterexample!**). (3 points each)

(a) The initial value problem $ay'' + by' + cy = 0$, $y(0) = 0$ has a unique solution for every $a, b, c \in \mathbb{R}$.

(b) If A is $n \times n$ and $A^2 = 0$ (zero matrix), then the characteristic polynomial of A is λ^n.

(c) If A can be row reduced to B, then A and B have the same determinant.

(d) If A is 5×6 and $\dim \operatorname{Nul} A = 1$, then $T(\vec{x}) = A\vec{x}$ is onto.

(e) Every linearly dependent subset of a finite dimensional vector space V has a subset that spans V.

Problem 4. (a) Let $f(x) = x^2 + 1$, defined on the interval $[0, \pi]$. Find the Fourier sine series of f, and graph the function it converges to on the interval $[-\pi, \pi]$. Be sure to indicate on the graph the value of the sine series at all points in $[-\pi, \pi]$. You do not have to evaluate any integrals. Your answer may have coefficients c_n in it, along with formula(s) $c_n = \cdots \int_a^b \cdots dx$. (6 points)

(b) Find a formal solution to the heat problem given below. Again, you do not need to evaluate any integrals, but you should provide the formula for any fundamental solutions u_n that you use. (4 points)

$$\begin{cases} u_t = 3u_{xx} & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0 & t > 0, \\ u(x,0) = x^2 + 1 & 0 < x < \pi. \end{cases}$$

Problem 5. If u is a function of the variables x and t, consider the PDE $u_{xx} + u_t + u_{tt} = 0$.

(a) If $u(x,t) = X(x)T(t)$ solves the PDE, derive ODEs (sharing a common constant) that X and T would have to satisfy. (4 points)

(b) Use (a) to give an example of a non-trivial solution to the PDE. (6 points)

Problem 6. Let V be the vector space of real 2×2 matrices, and define the function $\mathrm{tr} : V \to \mathbb{R}$ by

$$\mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

That is, the tr function outputs the sum of the entries on the diagonal. Define an inner product on V by $\langle A, B\rangle = \mathrm{tr}(AB^t)$. You may assume this has all of the properties of an inner product except the ones you are asked to prove in part (a).

(a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Show that $\langle A, B\rangle = \langle B, A\rangle$. Also show that if $\langle A, A\rangle = 0$, then A is the zero matrix. (4 points)

(b) Prove that $|ax + by + cz + dw| \le \sqrt{(a^2 + b^2 + c^2 + d^2)(x^2 + y^2 + z^2 + w^2)}$ for any real numbers x, y, z, w, a, b, c, d. (5 points)

(c) If $W = \left\{\begin{bmatrix} a & b \\ b & d \end{bmatrix} : a, b, d \in \mathbb{R}\right\}$ is the subspce of V consisting of symmetric matrices, find a basis for $W^\perp$ (with respect to the given inner product). (6 points)

Problem 7. Let

$$A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

where x_{ij} is an **integer** for every i, j. Assume that all of the diagonal entries in A are odd, and that all of the non-diagonal entries are even. That is, x_{ij} is odd if and only if $i = j$. Is A always invertible, sometimes invertible, or never invertible? (5 points)